Gage & Co's.
Educational Series

# GEOMETRY.

Hamblin Smith.

OT
513
Ont.
1886 S
Bks. I-III

Presented to
The Library of
The Ontario College of
Education
The University of Toronto
by
R.A. Paterson, B.A

516
1886
G136
S651E
Bk.1-3
UTC

ONTARIO COLLEGE OF
EDUCATION
~~513.8~~
UNIVERSITY OF TORONTO

R. S. CORMACK,
BOOKS AND STATIONERY
WHITBY, - - - ONT.

A C D B [illegible] $CA \cdot CB = \frac{1}{4} AB$ [illegible]

[illegible]

Then [illegible] $CB \cdot CA = \frac{1}{4}$ [illegible]

$= \frac{1}{4}(AB$ [illegible]$)$

VI B A D B C [illegible] $AC \cdot CB$, $\overline{BC}^2 =$ [illegible]

[illegible]

Then $BA =$ [illegible] $BC + AB$ [illegible]

[illegible] $= \frac{1}{2}$ [illegible]

$= \frac{1}{4}(BC + DC)^2$

[illegible]

W. J. Gage & Co'.s Mathematical Series.

# ELEMENTS OF GEOMETRY.

CONTAINING

## BOOKS I. TO III.

WITH

*EXERCISES AND NOTES*

BY

J. HAMBLIN SMITH, M. A.,

*Of Gonville and Caius College, and late Lecturer at St. Peter's College, Cambridge.*

*Prescribed by the Council of Public Instruction for use in the Schools of Nova Scotia.*

*Authorized for use in the Schools of Manitoba.*

*Recommended by the University of Halifax, Nova Scotia.*

*Recommended by the Council of Public Instruction, Quebec.*

*Authorized by the Education Department, Ontario.*

1886.

W. J. GAGE & COMPANY,

TORONTO.

[illegible]

$\overline{aE}^2 + \overline{AE}^2 = 2(\overline{aC}^2 + \overline{CE}^2)$ [illegible]

∴ $(aC + cB)^2 + (cE - [illegible])^2 = 2(\overline{aC}^2 + \overline{cE}^2)$

∴ $(aD + aC)^2 + (cB - aC)^2 = 2(\overline{cB}^2 + \overline{aC}^2)$,

[illegible]

∴ [illegible]

350

Digitized by the Internet Archive
in 2008 with funding from
Microsoft Corporation

http://www.archive.org/details/elementsofgeome00smituoft

# PREFACE.

To preserve Euclid's order, to supply omissions, to remove defects, to give brief notes of explanation and simpler methods of proof in cases of acknowledged difficulty—such are the main objects of this Edition of the Elements.

The work is based on the Greek text, as it is given in the Editions of August and Peyrard. To the suggestions of the late Professor De Morgan, published in the Companion to the British Almanack for 1849, I have paid constant deference.

A limited use of symbolic representation, wherein the symbols stand for words and not for operations, is generally regarded as desirable, and I have been assured, by the highest authorities on this point, that the symbols employed in this book are admissible in the Examinations at Oxford and Cambridge.[1]

I have generally followed Euclid's method of proof, but not to the exclusion of other methods recommended by their simplicity, such as the demonstrations by which I propose to replace (at least for a first reading) the difficult Theorems 5 and 7 in the First Book. I have also attempted to render many of the proofs, as for instance Propositions 2, 13, and 35 in Book I., and Proposition 13 in Book II., less confusing to the learner.

---

[1] I regard this point as completely settled in Cambridge by the following notices prefixed to the papers on Euclid set in the Senate-House Examinations:

I. In the Previous Examination:

*In answers to these questions any intelligible symbols and abbreviations may be used.*

II. In the Mathematical Tripos:

*In answers to the questions on Euclid the symbol — must not be used. The only abbreviation admitted for the square on* AB *is* "*sq. on* AB," *and for the rectangle contained by* AB *and* CD, "*rect.* AB, CD."

In Propositions 4, 5, 6, 7, and 8 of the Second Book I have ventured to make an important change in Euclid's mode of exposition, by omitting the diagonals from the diagrams and the gnomons from the text.

In the Third Book I have deviated with even greater boldness from the precise line of Euclid's method. For it is in treating of the properties of the circle that the importance of certain matters, to which reference is made in the Notes of the present volume, is fully brought out. I allude especially to the application of Superposition as a test of equality, to the conception of an Angle as a magnitude capable of unlimited increase, and to the development of the methods connected with Loci and Symmetry.

The Exercises have been selected with considerable care, chiefly from the Senate House Examination Papers. They are intended to be progressive and easy, so that a learner may from the first be induced to work out something for himself.

I desire to express my thanks to the friends who have improved this work by their suggestions, and to beg for further help of the same kind.

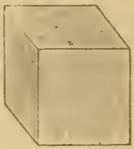
J. HAMBLIN SMITH

CAMBRIDGE, 1873.

# ELEMENTS OF GEOMETRY.

## INTRODUCTORY REMARKS.

When a block of stone is hewn from the rock, we call it a Solid *Body*. The stone-cutter shapes it, and brings it into that which we call *regularity of form;* and then it becomes a Solid *Figure.*



Now suppose the figure to be such that the block has six flat sides, each the exact counterpart of the others; so that, to one who stands facing a corner of the block, the three sides which are visible present the appearance represented in this diagram.

Each side of the figure is called a *Surface;* and when smoothed and polished, it is called a *Plane* Surface.

The sharp and well-defined edges, in which each pair of sides meets, are called *Lines.*

The place, at which any three of the edges meet, is called a *Point.*

A *Magnitude* is anything which is made up of parts in any way like itself. Thus, a line is a magnitude; because we may regard it as made up of parts which are themselves lines.

The properties Length, Breadth (or Width), and Thickness (or Depth or Height) of a body are called its *Dimensions.*

We make the following distinction between Solids, Surfaces Lines, and Points:

A Solid has three dimensions, Length, Breadth, Thickness.

A Surface has two dimensions, Length, Breadth.

A Line has one dimension, Length.

A point has no dimensions.

# BOOK I.

## DEFINITIONS.

I. A Point is that which has no parts.

This is equivalent to saying that a Point has no magnitude, since we define it as that which cannot be divided into smaller parts.

II. A Line is length without breadth.

We cannot conceive a visible line without breadth ; but we can reason about lines as if they had no breadth, and this is what Euclid requires us to do.

III. The Extremities of finite Lines are points.

A point marks *position*, as for instance, the place where a line begins or ends, or meets or crosses another line.

IV. A Straight Line is one which lies in the same direction from point to point throughout its length.

V. A Surface is that which has length and breadth only.

VI. The Extremities of a Surface are lines.

VII. A Plane Surface is one in which, if any two points be taken, the straight line between them lies wholly in that surface.

Thus the ends of an uncut cedar-pencil are plane surfaces ; but the rest of the surface of the pencil is not a plane surface, since two points may be taken in it such that the *straight* line joining them will not lie on the surface of the pencil.

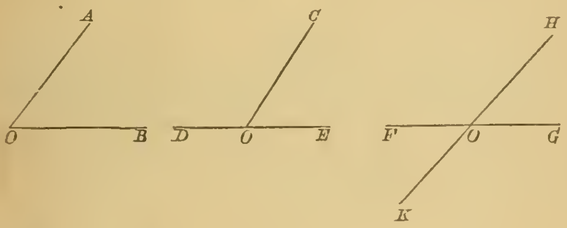
In our introductory remarks we gave examples of a Surface, a Line, and a Point, as we know them through the evidence of the senses

The Surfaces, Lines, and Points of Geometry may be regarded as mental pictures of the surfaces, lines, and points which we know from experience.

It is, however, to be observed that Geometry requires us to conceive the possibility of the existence
of a Surface apart from a Solid body,
of a Line apart from a Surface,
of a Point apart from a Line.

VIII. When two straight lines meet one another, the inclination of the lines to one another is called an Angle.

When *two* straight lines have one point common to both, they are said to *form* an angle (or angles) at that point. The point is called the *vertex* of the angle (or angles), and the lines are called the *arms* of the angle (or angles).

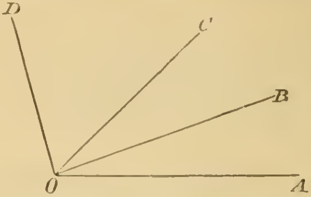


Thus, if the lines $OA$, $OB$ are terminated at the same point $O$, they form an angle, which is called *the angle at* $O$, or *the angle* $AOB$, or *the angle* $BOA$,—the letter which marks the vertex being put between those that mark the arms.

Again, if the line $CO$ meets the line $DE$ at a point in the line $DE$, so that $O$ is a point common to both lines, $CO$ is said to make with $DE$ the angles $COD$, $COE$; and these (as having one arm, $CO$, common to both) are called *adjacent* angles.

Lastly, if the lines $FG$, $HK$ cut each other in the point $O$, the lines make with each other four angles $FOH$, $HOG$, $GOK$, $KOF$; and of these $GOH$, $FOK$ are called *vertically opposite* angles, as also are $FOH$ and $GOK$.

When *three or more* straight lines as *OA*, *OB*, *OC*, *OD* have a point *O* common to all, the angle formed by one of them, *OD*, with *OA* may be regarded as being made up of the angles *AOB*, *BOC*, *COD*; that is, we may speak of the angle *AOD* as a whole, of which the parts are the angles *AOB*, *BOC*, and *COD*.

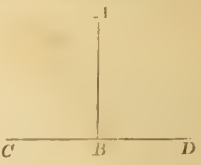


Hence we may regard an angle as a *Magnitude*, inasmuch as any angle may be regarded as being made up of parts which are themselves angles.

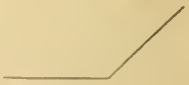
The size of an angle depends in no way on the length of the arms by which it is bounded.

We shall explain hereafter the restriction on the magnitude of angles enforced by Euclid's definition, and the important results that follow an extension of the definition.

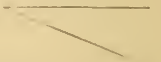
IX. When a straight line (as *AB*) meeting another straight line (as *CD*) makes the adjacent angles (*ABC* and *ABD*) equal to one another, each of the angles is called a RIGHT ANGLE; and each line is said to be a PERPENDICULAR to the other.



X. An OBTUSE ANGLE is one which is greater than a right angle.



XI. An ACUTE ANGLE is one which is less than a right angle.

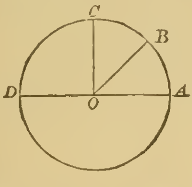


XII. A FIGURE is that which is enclosed by one or more boundaries.

XIII. A Circle is a plane figure contained by one line, which is called the Circumference, and is such, that all straight lines drawn to the circumference from a certain point (called the Centre) within the figure are equal to one another.

XIV. Any straight line drawn from the centre of a circle to the circumference is called a Radius.

XV. A Diameter of a circle is a straight line drawn through the centre and terminated both ways by the circumference.



Thus, in the diagram, $O$ is the centre of the circle $ABCD$, $OA$. $OB$, $OC$, $OD$ are Radii of the circle, and the straight line $AOD$ is a Diameter. Hence the radius of a circle is half the diameter.

XVI. A Semicircle is the figure contained by a diameter and the part of the circumference cut off by the diameter.

XVII. Rectilinear figures are those which are contained by straight lines.

The Perimeter (or Periphery) of a rectilinear figure is the sum of its sides.

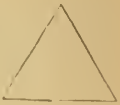
XVIII. A Triangle is a plane figure contained by three straight lines.

XIX. A Quadrilateral is a plane figure contained by four straight lines.

XX. A Polygon is a plane figure contained by more than four straight lines.

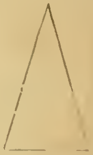
When a polygon has all its sides equal and all its angles equal it is called a *regular* polygon.

XXI. An EQUILATERAL Triangle is one which has all its sides equal.



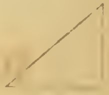
XXII. An ISOSCELES Triangle is one which has two sides equal.

The third side is often called the *base* of the triangle.



The term *base* is applied to any one of the sides of a triangle to distinguish it from the other two, especially when they have been previously mentioned.

XXIII. A RIGHT-ANGLED Triangle is one in which one of the angles is a right angle.

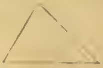


The side *subtending*, that is, *which is opposite* the right angle, is called the *Hypotenuse*.

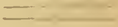
XXIV. An OBTUSE-ANGLED Triangle is one in which one of the angles is obtuse.

It will be shewn hereafter that a triangle can have only one of its angles either equal to, or greater than, a right angle.

XXV. An ACUTE-ANGLED Triangle is one in which ALL the angles are acute.



XXVI. PARALLEL STRAIGHT LINES are such as, being in the same plane, never meet when continually produced in both directions.



Euclid proceeds to put forward Six Postulates, or Requests, that he may be allowed to make certain assumptions on the construction of figures and the properties of geometrical magnitudes.

## POSTULATES

Let it be granted—

I. That a straight line may be drawn from any one point to any other point.

II. That a terminated straight line may be produced to any length in a straight line.

III. That a circle may be described from any centre at any distance from that centre.

IV. That all right angles are equal to one another.

V. That two straight lines cannot enclose a space.

VI. That if a straight line meet two other straight lines, so as to make the two interior angles on the same side of it, taken together, less than two right angles, these straight lines being continually produced shall at length meet upon that side, on which are the angles, which are together less than two right angles.

The word rendered "Postulates" is in the original *αἰτήματα*, "requests."

In the first three Postulates Euclid states the use, under certain restrictions, which he desires to make of certain instruments for the construction of lines and circles.

In Post. I. and II. he asks for the use of the straight ruler, wherewith to draw straight lines. The restriction is, that the ruler is not supposed to be marked with divisions so as to measure lines.

In Post. III. he asks for the use of a pair of compasses, wherewith to describe a circle, whose centre is at one extremity of a given line, and whose circumference passes through the other extremity of that line. The restriction is, that the compasses are not supposed to be capable of conveying distances.

Post. IV. and V. refer to simple geometrical facts, which Euclid desires to take for granted.

Post. VI. may, as we shall shew hereafter, be deduced from a more simple Postulate. The student must defer the consideration of this Postulate, till he has reached the 17th Proposition of Book I.

Euclid next enumerates, as statements of fact, nine Axioms or, as he calls them, Common Notions, applicable (with the exception of the eighth) to all kinds of magnitudes, and not necessarily restricted, as are the Postulates, to *geometrical* magnitudes.

### AXIOMS.

I. Things which are equal to the same thing are equal to one another.

II. If equals be added to equals, the wholes are equal.

III. If equals be taken from equals, the remainders are equal.

IV. If equals and unequals be added together, the wholes are unequal.

V. If equals be taken from unequals, or unequals from equals, the remainders are unequal.

VI. Things which are double of the same thing, or of equal things, are equal to one another.

VII. Things which are halves of the same thing, or of equal things, are equal to one another.

VIII. Magnitudes which coincide with one another are equal to one another.

IX. The whole is greater than its part.

With his Common Notions Euclid takes the ground of authority, saying in effect, "To my Postulates I request, to my Common Notions I claim, your assent."

Euclid develops the science of Geometry in a series of Propositions, some of which are called Theorems and the rest Problems, though Euclid himself makes no such distinction.

By the name *Theorem* we understand a truth, capablo of demonstration or proof by deduction from truths previously admitted or proved.

By the name *Problem* we understand a construction, capable of being effected by the employment of principles of construction previously admitted or proved.

A *Corollary* is a Theorem or Problem easily deduced from, or effected by means of, a Proposition to which it is attached.

We shall divide the First Book of the Elements into three sections. The reason for this division will appear in the course of the work.

## SYMBOLS AND ABBREVIATIONS USED IN BOOK I.

| | | | |
|---|---|---|---|
| ∵ *for* | because | ⊙ *for* | circle |
| ∴ | therefore | ○ce | circumference |
| = | is (or are) equal to | ∥ | parallel |
| ∠ | angle | ▱ | parallelogram |
| △ | triangle | ⊥ | perpendicular |

| | | | |
|---|---|---|---|
| equilat. | equilateral | reqd. | required |
| extr. | exterior | rt. | right |
| intr. | interior | sq. | square |
| pt | point | sqq. | squares |
| rectil. | rectilinear | st. | straight |

It is well known that one of the chief difficulties with learners of Euclid is to distinguish between what is assumed, or given, and what has to be proved in some of the Propositions. To make the distinction clearer we shall put in italics the statements of what has to be done in a Problem, and what has to be proved in a Theorem. The last line in the proof of every Proposition states, that what had to be done or proved has been done or proved.

The letters Q. E. F. at the end of a Problem stand for *Quod erat faciendum.*

The letters Q. E. D. at the end of a Theorem stand for *Quod erat demonstrandum.*

In the marginal references :

Post. stands for Postulate.
Def. ............... Definition.
Ax. ............... Axiom.
I. 1. ............... Book I. Proposition 1.

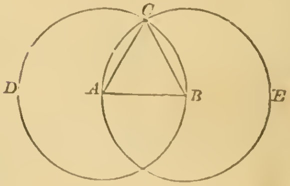
Hyp. stands for Hypothesis, *supposition*, and refers to something granted, or assumed to be true.

# SECTION I.

## *On the Properties of Triangles.*

### Proposition I. Problem.

*To describe an equilateral triangle on a given straight line.*



Let $AB$ be the given st. line.

*It is required to describe an equilat.* △ *on* $AB$

With centre $A$ and distance $AB$ describe ⊙ $BCD$. Post. 3.
With centre $B$ and distance $BA$ describe ⊙ $ACE$. Post. 3
From the pt. $C$, in which the ⊙s cut one another,
draw the st. lines $CA$, $CB$. Post. 1.
Then will $ABC$ be an equilat. △.

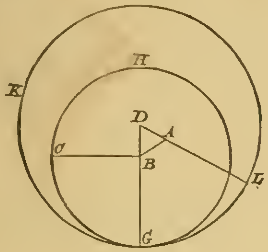
For ∵ $A$ is the centre of ⊙ $BCD$,
∴ $AC=AB$. Def. 13
And ∵ $B$ is the centre of ⊙ $ACE$,
∴ $BC=AB$. Def. 13.
Now ∵ $AC$, $BC$ are each $=AB$,
∴ $AC=BC$. Ax. 1.

Thus $AC$, $AB$, $BC$ are all equal, and an equilat. △ $ABC$ has been described on $AB$.

Q. E. F.

## PROPOSITION II. PROBLEM.

*From a given point to draw a straight line equal to a given straight line.*



Let $A$ be the given pt., and $BC$ the given st. line.

*It is required to draw from $A$ a st. line equal to $BC$.*

From $A$ to $B$ draw the st. line $AB$. Post. 1.

On $AB$ describe the equilat. △ $ABD$. I. 1.

With centre $B$ and distance $BC$ describe ⊙ $CGH$. Post. 3.

Produce $DB$ to meet the ○ce $CGH$ in $G$.

With centre $D$ and distance $DG$ describe ⊙ $GKL$. Post. 3.

Produce $DA$ to meet the ○ce $GKL$ in $L$.

Then will $AL=BC$.

For $\because B$ is the centre of ⊙ $CGH$,

$\therefore BC=BG$. Def. 13.

And $\because D$ is the centre of ⊙ $GKL$,

$\therefore DL=DG$. Def. 13.

And parts of these, $DA$ and $DB$, are equal. Def. 21

$\therefore$ remainder $AL$=remainder $BG$. Ax. 3.

But $BC=BG$;

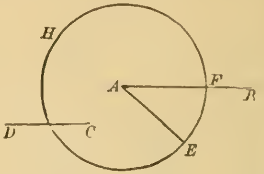
$\therefore AL=BC$. Ax. 1.

Thus from pt. $A$ a st. line $AL$ has been drawn $=BC$.

Q. E. F.

## PROPOSITION III. PROBLEM.

*From the greater of two given straight lines to cut off a part equal to the less.*



Let $AB$ be the greater of the two given st. lines $AB$, $CD$.

*It is required to cut off from $AB$ a part $= CD$.*

From $A$ draw the st. line $AE = CD$. I. 2.

With centre $A$ and distance $AE$ describe ⊙ $EFH$, cutting $AB$ in $F$.

Then will $AF = CD$.

For $\because$ $A$ is the centre of ⊙ $EFH$,

$\therefore$ $AF = AE$.

But $AE = CD$;

$\therefore$ $AF = CD$. Ax. 1.

Thus from $AB$ a part $AF$ has been cut off $= CD$.

Q. E. F.

### EXERCISES.

1. Shew that if straight lines be drawn from $A$ and $B$ in the diagram of Prop. I. to the other point in which the circles intersect, another equilateral triangle will be described on $AB$.

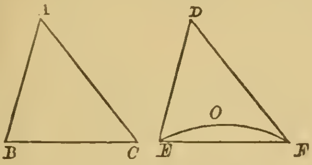
2. By a construction similar to that in Prop. III. produce the less of two given straight lines that it may be equal to the greater.

3. Draw a figure for the case in Prop. II., in which the given point coincides with $B$.

4. By a similar construction to that in Prop. I. describe on a given straight line an isosceles triangle, whose equal sides shall be each equal to another given straight line.

### PROPOSITION IV. THEOREM.

*If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise the angles contained by those sides equal to one another, they must have their third sides equal; and the two triangles must be equal, and the other angles must be equal, each to each, viz. those to which the equal sides are opposite.*



In the $\triangle$s $ABC$, $DEF$,
let $AB=DE$, and $AC=DF$, and $\angle\ BAC=\angle\ EDF$.

*Then must* $BC=EF$ *and* $\triangle\ ABC=\triangle\ DEF$, *and the other* $\angle$ *s, to which the equal sides are opposite, must be equal, that is,* $\angle\ ABC=\angle\ DEF$ *and* $\angle\ ACB=\angle\ DFE$.

For, if $\triangle\ ABC$ be applied to $\triangle\ DEF$,
so that $A$ coincides with $D$, and $AB$ falls on $DE$,
then $\because AB=DE$, $\therefore B$ will coincide with $E$.
And $\because AB$ coincides with $DE$, and $\angle\ BAC=\angle\ EDF$, Hyp.
$\therefore AC$ will fall on $DF$.
Then $\because AC=DF$, $\therefore C$ will coincide with $F$.
And $\because B$ will coincide with $E$, and $C$ with $F$,
$\therefore BC$ will coincide with $EF$;

for if not, let it fall otherwise as $EOF$: then the two st. lines $BC$, $EF$ will enclose a space, which is impossible. Post. 5.

$\therefore BC$ will coincide with and $\therefore$ is equal to $EF$, Ax. 8.

and $\triangle\ ABC$.................................. $\triangle\ DEF$,
and $\angle\ ABC$.................................. $\angle\ DEF$,
and $\angle\ ACB$.................................. $\angle\ DFE$.

Q. E. D.

## NOTE 1. *On the Method of Superposition.*

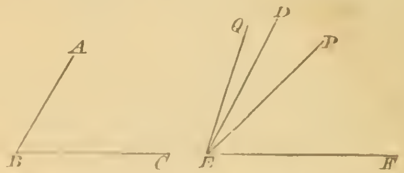
Two geometrical magnitudes are said, in accordance with Ax. VIII. to be *equal*, when they can be so placed that the boundaries of the one coincide with the boundaries of the other.

Thus, two straight lines are equal, if they can be so placed that the points at their extremities coincide: and two angles are equal, if they can be so placed that their vertices coincide in position and their arms in direction: and two triangles are equal, if they can be so placed that their sides coincide in direction and magnitude.

In the application of the test of equality by this *Method of Superposition*, we assume that an angle or a triangle may be moved from one place, turned over, and put down in another place, without altering the relative positions of its boundaries.

We also assume that if one part of a straight line coincide with one part of another straight line, the other parts of the lines also coincide in direction; or, that straight lines, which coincide in two points, coincide when produced.

The method of Superposition enables us also to compare magnitudes of the same kind that are unequal. For example, suppose $ABC$ and $DEF$ to be two given angles.



Suppose the arm $BC$ to be placed on the arm $EF$, and the vertex $B$ on the vertex $E$.

Then, if the arm $BA$ coincide in direction with the arm $ED$, the angle $ABC$ is equal to $DEF$.

If $BA$ fall between $ED$ and $EF$ in the direction $EP$, $ABC$ is less than $DEF$.

If $BA$ fall in the direction $EQ$ so that $ED$ is between $EQ$ and $EF$, $ABC$ is greater than $DEF$.

## Note 2. *On the Conditions of Equality of two Triangles.*

A Triangle is composed of six parts, three sides and three angles.

When the six parts of one triangle are equal to the six parts of another triangle, each to each, the Triangles are said to be equal in all respects.

There are four cases in which Euclid proves that two triangles are equal in all respects; viz., when the following parts are equal in the two triangles.

| | | |
|---|---|---|
| 1. | Two sides and the angle between them. | I. 4. |
| 2. | Two angles and the side between them. | I. 26. |
| 3. | The three sides of each. | I. 8. |
| 4. | Two angles and the side opposite one of them. | I. 26. |

The Propositions, in which these cases are proved, are the most important in our First Section.

The first case we have proved in Prop. IV.

Availing ourselves of the method of superposition, we can prove Cases 2 and 3 by a process more simple than that employed by Euclid, and with the further advantage of bringing them into closer connexion with Case 1. We shall therefore give three Propositions, which we designate A, B, and C, in the Place of Euclid's Props. V. VI. VII. VIII.

The displaced Propositions will be found on pp. 108-112.

Proposition A corresponds with Euclid I. 5.
.............. B .............................. I. 26, first part.
.............. C .............................. I. 8.

### PROPOSITION A. THEOREM.

*If two sides of a triangle be equal, the angles opposite those sides must also be equal.*

FIG. 1.

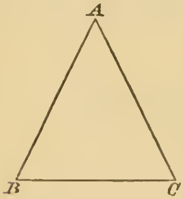
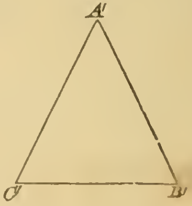


FIG. 2.



In the isosceles triangle $ABC$, let $AC=AB$. (Fig. 1.)

*Then must* $\angle ABC = \angle ACB$

Imagine the $\triangle ABC$ to be taken up, turned round, and set down again in a reversed position as in Fig. 2, and designate the angular points $A'$, $B'$, $C'$.

Then in $\triangle$s $ABC$, $A'C'B'$,

$\because AB=A'C'$, and $AC=A'B'$, and $\angle BAC = \angle C'A'B'$,

$\therefore \angle ABC = \angle A'C'B'$. I. 4.

But $\angle A'C'B' = \angle ACB$;

$\therefore \angle ABC = \angle ACB$. Ax. 1.

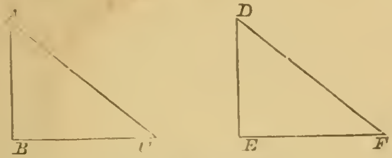
Q.E.D.

Cor. Hence every equilateral triangle is also equiangular.

Note. When one side of a triangle is distinguished from the other sides by being called the *Base*, the angular point opposite to that side is called the *Vertex* of the triangle.

## PROPOSITION B. THEOREM.

*If two triangles have two angles of the one equal to two angles of the other, each to each, and the sides adjacent to the equal angles in each also equal; then must the triangles be equal in all respects.*



In △s $ABC$, $DEF$,

let ∠ $ABC=$ ∠ $DEF$, and ∠ $ACB=$ ∠ $DFE$, and $BC=EF$.

*Then must* $AB=DE$, *and* $AC=DF$, *and* ∠ $BAC=$ ∠ $EDF$.

For if △ $DEF$ be applied to △ $ABC$, so that $E$ coincides with $B$, and $EF$ falls on $BC$;

then ∵ $EF=BC$, ∴ $F$ will coincide with $C$;

and ∵ ∠ $DEF=$ ∠ $ABC$, ∴ $ED$ will fall on $BA$;

∴ $D$ will fall on $BA$ or $BA$ produced.

Again, ∵ ∠ $DFE=$ ∠ $ACB$, ∴ $FD$ will fall on $CA$;

∴ $D$ will fall on $CA$ or $CA$ produced.

∴ $D$ must coincide with $A$, the only pt. common to $BA$ and $CA$.

∴ $DE$ will coincide with and ∴ is equal to $AB$,

and $DF$.......................................... $AC$,

and ∠ $EDF$.......................................... ∠ $BAC$,

and △ $DEF$.......................................... △ $ABC$;

and ∴ the triangles are equal in all respects.

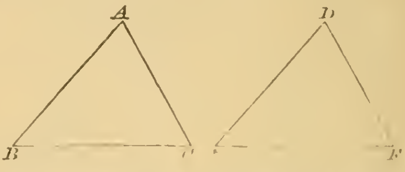
Q. E. D.

COR. Hence, by a process like that in Prop. A, we can prove the following theorem:

*If two angles of a triangle be equal the sides which subtend them are also equal.* (Eucl. I. 6.)

### PROPOSITION C. THEOREM.

*If two triangles have the three sides of the one equal to the three sides of the other, each to each, the triangles must be equal in all respects.*

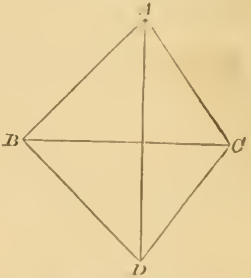


Let the three sides of the $\triangle$s $ABC$, $DEF$ be equal, each to each, that is, $AB=DE$, $AC=DF$, and $BC=EF$.

*Then must the triangles be equal in all respects.*

Imagine the $\triangle DEF$ to be turned over and applied to the $\triangle ABC$, in such a way that $EF$ coincides with $BC$, and the vertex $D$ falls on the side of $BC$ opposite to the side on which $A$ falls; and join $AD$.

CASE I. When $AD$ passes through $BC$.



Then in $\triangle ABD$, $\because BD=BA$, $\therefore \angle BAD=\angle BDA$, I. A.

And in $\triangle ACD$, $\because CD=CA$, $\therefore \angle CAD=\angle CDA$, I. A.

$\therefore$ sum of $\angle$s $BAD$, $CAD$ = sum of $\angle$s $BDA$, $CDA$, Ax. 2.

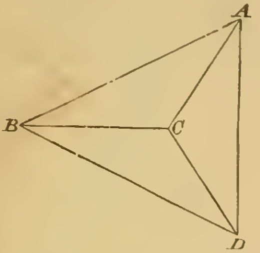
that is, $\angle BAC=\angle BDC$.

Hence we see, referring to the original triangles, that

$$\angle BAC=\angle EDF.$$

$\therefore$, by Prop. 4, the triangles are equal in all respects.

Case II. When the line joining the vertices does not pass through $BC$.



Then in $\triangle ABD$, $\because BD = BA$, $\therefore \angle BAD = \angle BDA$, I. A.

And in $\triangle ACD$, $\because CD = CA$, $\therefore \angle CAD = \angle CDA$, I. A.

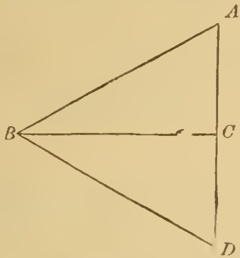
Hence since the whole angles $BAD$, $BDA$ are equal.

and parts of these $CAD$, $CDA$ are equal.

$\therefore$ the remainders $BAC$, $BDC$ are equal. Ax. 3.

Then, as in Case I., the equality of the original triangles may be proved.

Case III. When $AC$ and $CD$ are in the same straight line.



Then in $\triangle ABD$, $\because BD = BA$, $\therefore \angle BAD = \angle BDA$, I. A.

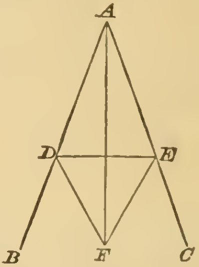
that is, $\angle BAC = \angle BDC$.

Then, as in Case I., the equality of the original triangles may be proved.

Q. E. D.

## PROPOSITION IX. PROBLEM.

*To bisect a given angle.*



Let $BAC$ be the given angle.

*It is required to bisect* $\angle BAC$.

In $AB$ take any pt. $D$.

In $AC$ make $AE=AD$, and join $DE$.

On $DE$, on the side remote from $A$, describe an equilat. $\triangle DFE$. I. 1.

Join $AF$. Then $AF$ will bisect $\angle BAC$.

For in $\triangle$s $AFD$, $AFE$,

$\because AD=AE$, and $AF$ is common, and $FD=FE$,

$\therefore \angle DAF = \angle EAF$, I. c.

that is, $\angle BAC$ is bisected by $AF$.

Q. E. F.

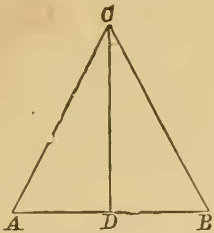
Ex. 1. Shew that we can prove this Proposition by means of Prop. IV. and PROP. A., without applying Prop. C.

Ex. 2. If the equilateral triangle, employed in the construction, be described with its vertex towards the given angle; shew that there is one case in which the construction will fail, and two in which it will hold good.

NOTE.—The line dividing an angle into two equal parts is called the BISECTOR of the angle.

### Proposition X. Problem.

*To bisect a given finite straight line.*



Let $AB$ be the given st. line.

*It is required to bisect $AB$.*

On $AB$ describe an equilat. △ $ACB$. I. 1.

Bisect ∠ $ACB$ by the st. line $CD$ meeting $AB$ in $D$; I. 9.

then $AB$ shall be bisected in $D$.

For in △s $ACD$, $BCD$,

∵ $AC=BC$, and $CD$ is common, and ∠ $ACD=$ ∠ $BCD$,

∴ $AD=BD$; I. 4.

∴ $AB$ is bisected in $D$.

Q. E. F.

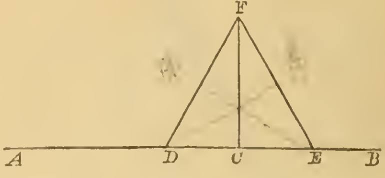
Ex. 1. The straight line, drawn to bisect the vertical angle of an isosceles triangle, also bisects the base.

Ex. 2. The straight line, drawn from the vertex of an isosceles triangle to bisect the base, also bisects the vertical angle.

Ex. 3. Produce a given finite straight line to a point, such that the part produced may be one-third of the line, which is made up of the whole and the part produced.

### PROPOSITION XI. PROBLEM.

*To draw a straight line at right angles to a given straight line from a given point in the same.*



Let $AB$ be the given st. line, and $C$ a given pt. in it.

*It is required to draw from $C$ a st. line $\perp$ to $AB$.*

Take any pt. $D$ in $AC$, and in $CB$ make $CE=CD$.

On $DE$ describe an equilat. $\triangle$ $DFE$. I. 1

Join $FC$. $FC$ shall be $\perp$ to $AB$.

For in $\triangle$s $DCF$, $ECF$,

$\because DC=CE$, and $CF$ is common, and $FD=FE$,

$\therefore \angle DCF = \angle ECF$, I. c.

and $\therefore FC$ is $\perp$ to $AB$. Def. 9.

Q. E. F.

COR. To draw a straight line at right angles to a given straight line $AC$ from one extremity, $C$, take any point $D$ in $AC$, produce $AC$ to $E$, making $CE=CD$, and proceed as in the proposition.

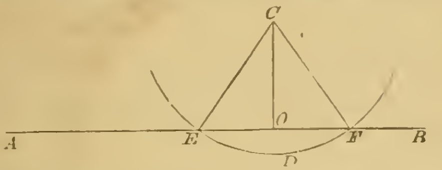
Ex. 1. Shew that in the diagram of Prop. IX. $AF$ and $ED$ intersect each other at right angles, and that $ED$ is bisected by $AF$.

Ex. 2. If $O$ be the point in which two lines, bisecting $AB$ and $AC$, two sides of an equilateral triangle, at right angles, meet; shew that $OA$, $OB$, $OC$ are all equal.

Ex. 3. Shew that Prop. XI. is a particular case of Prop. IX.

## PROPOSITION XII. PROBLEM.

*To draw a straight line perpendicular to a given straight line of an unlimited length from a given point without it.*



Let $AB$ be the given st. line of unlimited length; $C$ the given pt. without it.

*It is required to draw from C a st. line ⊥ to AB.*

Take any pt. $D$ on the other side of $AB$.

With centre $C$ and distance $CD$ describe a ⊙ cutting $AB$ in $E$ and $F$.

Bisect $EF$ in $O$, and join $CE$, $CO$, $CF$. I. 10

Then $CO$ shall be ⊥ to $AB$.

For in △s $COE$, $COF$,

∵ $EO=FO$, and $CO$ is common, and $CE=CF$,

∴ $\angle COE=\angle COF$; I. c.

∴ $CO$ is ⊥ to $AB$. Def. 9.

Q. E. F.

Ex. 1. If the straight line were not of unlimited length, how might the construction fail?

Ex. 2. If in a triangle the perpendicular from the vertex on the base bisect the base, the triangle is isosceles.

Ex. 3. The lines drawn from the angular points of an equilateral triangle to the middle points of the opposite sides are equal.

### *Miscellaneous Exercises on Props. I. to XII.*

1. Draw a figure for Prop. II. for the case when the given point $A$ is

(α) below the line $BC$ and to the right of it.

(β) below the line $BC$ and to the left of it.

2. Divide a given angle into four equal parts.

3. The angles $B$, $C$, at the base of an isosceles triangle, are bisected by the straight lines $BD$, $CD$, meeting in $D$; shew that $BDC$ is an isosceles triangle.

4. $D$, $E$, $F$ are points taken in the sides $BC$, $CA$, $AB$, of an equilateral triangle, so that $BD=CE=AF$. Shew that the triangle $DEF$ is equilateral.

5. In a given straight line find a point equidistant from two given points; 1st, on the same side of it; 2d, on opposite sides of it.

6. $ABC$ is a triangle having the angle $ABC$ acute. In $BA$, or $BA$ produced, find a point $D$ such that $BD=CD$.

7. The equal sides $AB$, $AC$, of an isosceles triangle $ABC$ are produced to points $F$ and $G$, so that $AF=AG$. $BG$ and $CF$ are joined, and $H$ is the point of their intersection. Prove that $BH=CH$, and also that the angle at $A$ is bisected by $AH$.

8. $BAC$, $BDC$ are isosceles triangles, standing on opposite sides of the same base $BC$. Prove that the straight line from $A$ to $D$ bisects $BC$ at right angles.

9. In how many directions may the line $AE$ be drawn in Prop. III.?

10. The two sides of a triangle being produced, if the angles on the other side of the base be equal, shew that the triangle is isosceles.

11. $ABC$, $ABD$ are two triangles on the same base $AB$ and on the same side of it, the vertex of each triangle being outside the other. If $AC=AD$, shew that $BC$ cannot $=BD$.

12. From $C$ any point in a straight line $AB$, $CD$ is drawn at right angles to $AB$, meeting a circle described with centre $A$ and distance $AB$ in $D$; and from $AD$, $AE$ is cut off $=AC$: shew that $AEB$ is a right angle.

## Proposition XIII. Theorem.

*The angles which one straight line makes with another upon one side of it are either two right angles, or together equal to two right angles.*

Fig. 1.

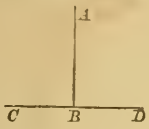
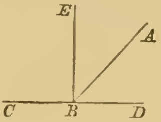


Fig. 2.



Let $AB$ make with $CD$ upon one side of it the ∠ s $ABC$, $ABD$.

*Then must these be either two rt. ∠ s,*
*or together equal to two rt. ∠ s*

First, if ∠ $ABC=$ ∠ $ABD$ as in Fig. 1,

each of them is a rt. ∠ . Def. 9.

Secondly, if ∠ $ABC$ be not $=$ ∠ $ABD$, as in Fig. 2,

from $B$ draw $BE$ ⊥ to $CD$. I. 11.

Then sum of ∠ s $ABC$, $ABD=$sum of ∠ s $EBC$, $EBA$, $ABD$,

and sum of ∠ s $EBC$, $EBD=$sum of ∠ s $EBC$, $EBA$, $ABD$;

∴ sum of ∠ s $ABC$, $ABD=$sum of ∠ s $EBC$, $EBD$;

Ax. 1.

∴ sum of ∠ s $ABC$, $ABD=$sum of a rt. ∠ and a rt. ∠ ;

∴ ∠ s $ABC$, $ABD$ are together $=$ two rt. ∠ s.

Q. E. D.

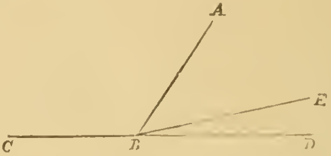
Ex. Straight lines drawn connecting the opposite angular points of a quadrilateral figure intersect each other in $O$. Shew that the angles at $O$ are together equal to four right angles.

Note (1.) If two angles together make up a right angle, each is called the Complement of the other. Thus, in fig. 2. ∠ $ABD$ is the complement of ∠ $ABE$.

(2.) If two angles together make up two right angles, each is called the Supplement of the other. Thus, in both figures, ∠ $ABD$ is the supplement of ∠ $ABC$.

## PROPOSITION XIV. THEOREM.

*If, at a point in a straight line, two other straight lines, upon the opposite sides of it, make the adjacent angles together equal to two right angles, these two straight lines must be in one and the same straight line.*



At the pt. $B$ in the st. line $AB$ let the st. lines $BC$, $BD$, on opposite sides of $AB$, make $\angle$s $ABC$, $ABD$ together = two rt. angles.

*Then $BD$ must be in the same st. line with $BC$.*

For if not, let $BE$ be in the same st. line with $BC$.

Then $\angle$s $ABC$, $ABE$ together = two rt. $\angle$s. I. 13.

And $\angle$s $ABC$, $ABD$ together = two rt. $\angle$s. Hyp

$\therefore$ sum of $\angle$s $ABC$, $ABE$ = sum of $\angle$s $ABC$, $ABD$.

Take away from each of these equals the $\angle ABC$;

then $\angle ABE = \angle ABD$, Ax. 3.

that is, the less = the greater; which is impossible,

$\therefore$ $BE$ is not in the same st. line with $BC$.

Similarly it may be shewn that no other line but $BD$ is in the same st. line with $BC$.

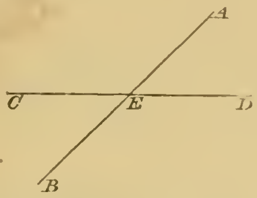
$\therefore$ $BD$ is in the same st. line with $BC$.

Q. E. D.

Ex. Shew the necessity of the words *the opposite sides* in the enunciation.

### PROPOSITION XV. THEOREM.

*If two straight lines cut one another, the vertically opposite angles must be equal.*



Let the st. lines $AB$, $CD$ cut one another in the pt. $E$.

*Then must* $\angle AEC = \angle BED$ *and* $\angle AED = \angle BEC$.

For $\because AE$ meets $CD$,

$\therefore$ sum of $\angle$ s $AEC$, $AED$ = two rt. $\angle$ s. I. 13.

And $\because DE$ meets $AB$,

$\therefore$ sum of $\angle$ s $BED$, $AED$ = two rt. $\angle$ s; I. 13.

$\therefore$ sum of $\angle$ s $AEC$, $AED$ = sum of $\angle$ s $BED$, $AED$;

$\therefore \angle AEC = \angle BED$. Ax. 3.

Similarly it may be shewn that $\angle AED = \angle BEC$.

Q. E. D.

Corollary I. From this it is manifest, that if two straight lines cut one another, the four angles, which they make at the point of intersection, are together equal to four right angles.

Corollary II. All the angles, made by any number of straight lines meeting in one point, are together equal to four right angles.

Ex. 1. Shew that the bisectors of $AED$ and $BEC$ are in the same straight line.

Ex. 2. Prove that $\angle AED$ is equal to the angle between two straight lines drawn at right angles from $E$ to $AE$ and $EC$, if both lie above $CD$.

Ex. 3. If $AB$, $CD$ bisect each other in $E$; shew that the triangles $AED$, $BEC$ are equal in all respects.

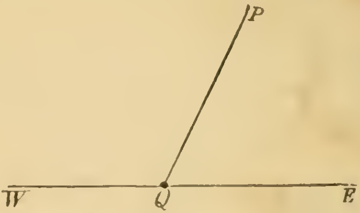
## Note 3. *On Euclid's definition of an Angle.*

Euclid directs us to regard an angle as the inclination of two straight lines to each other, which meet, *but are not in the same straight line.*

Thus he does not recognise the existence of a single angle equal in magnitude to two right angles.

The words printed in italics are omitted as needless, in Def. VIII., p. 3, and that definition may be extended with advantage in the following terms :—

Def. Let $WQE$ be a fixed straight line, and $QP$ a line which revolves about the fixed point $Q$, and which at first coincides with $QE$.



Then, when $QP$ has reached the position represented in the diagram, we say that it has described the angle $EQP$.

When $QP$ has revolved so far as to coincide with $QW$, we say that it has described an angle *equal to two right angles.*

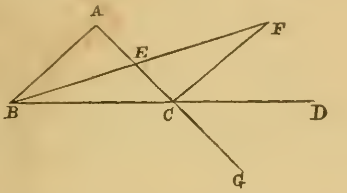
Hence we may obtain an easy proof of Prop. XIII. ; for whatever the position of $PQ$ may be, the angles which it makes with $WE$ are together equal to two right angles.

Again, in Prop. XV. it is evident that $\angle AED = \angle BEC$, since each has the same supplementary $\angle AEC$.

We shall shew hereafter, p. 149, how this definition may be extended, so as to embrace angles *greater than two right angles.*

## PROPOSITION XVI. THEOREM.

*If one side of a triangle be produced, the exterior angle is greater than either of the interior opposite angles.*



Let the side $BC$ of $\triangle ABC$ be produced to $D$.

*Then must $\angle ACD$ be greater than either $\angle CAB$ or $\angle ABC$.*

Bisect $AC$ in $E$, and join $BE$. I. 10.

Produce $BE$ to $F$, making $EF=BE$, and join $FC$.

Then in $\triangle$s $BEA$, $FEC$,

$\because BE=FE$, and $EA=EC$, and $\angle BEA=\angle FEC$, I. 15.

$\therefore \angle ECF=\angle EAB$. I. 4.

Now $\angle ACD$ is greater than $\angle ECF$; Ax. 9

$\therefore \angle ACD$ is greater than $\angle EAB$,

that is, $\angle ACD$ is greater than $\angle CAB$.

Similarly, if $AC$ be produced to $G$ it may be shewn that

$\angle BCG$ is greater than $\angle ABC$.

and $\angle BCG=\angle ACD$; I. 15

$\therefore \angle ACD$ is greater than $\angle ABC$.

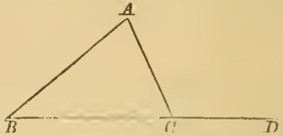
Q. E. D.

Ex. 1. From the same point there cannot be drawn more than two equal straight lines to meet a given straight line.

Ex. 2. If, from any point, a straight line be drawn to a given straight line making with it an acute and an obtuse angle, and if, from the same point, a perpendicular be drawn to the given line; the perpendicular will fall on the side of the acute angle.

### PROPOSITION XVII. THEOREM.

*Any two angles of a triangle are together less than two right angles.*



Let $ABC$ be any $\triangle$.

*Then must any two of its* $\angle$*s be together less than two rt.* $\angle$*s.*

Produce $BC$ to $D$.

Then $\angle ACD$ is greater than $\angle ABC$. I. 16.

$\therefore$ $\angle$s $ACD$, $ACB$ are together greater than $\angle$s $ABC$, $ACB$.

But $\angle$s $ACD$, $ACB$ together = two rt. $\angle$s. I. 13

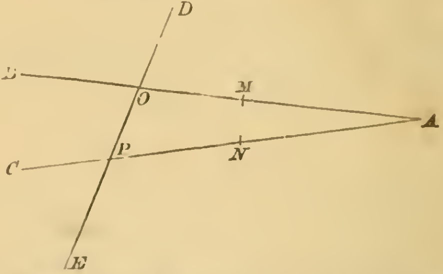
$\therefore$ $\angle$s $ABC$, $ACB$ are together less than two rt. $\angle$s.

Similarly it may be shewn that $\angle$s $ABC$, $BAC$ and also that $\angle$s $BAC$, $ACB$ are together less than two rt. $\angle$s.

Q. E. D.

### NOTE 4. *On the Sixth Postulate.*

We learn from Prop. XVII. that if two straight lines $BM$ and $CN$, which meet in $A$, are met by another straight line $DE$ in the points $O$, $P$,

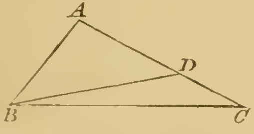


the angles $MOP$ and $NPO$ are together less than two right angles.

The Sixth Postulate asserts that if a line $DE$ meeting two other lines $BM$, $CN$ makes $MOP$, $NPO$, the two interior angles on the same side of it, together less than two right angles, $BM$ and $CN$ shall meet if produced on the same side of $DE$ on which are the angles $MOP$ and $NPO$.

## PROPOSITION XVIII. THEOREM.

*If one side of a triangle be greater than a second, the angle opposite the first must be greater than that opposite the second.*



In $\triangle ABC$, let side $AC$ be greater than $AB$.

*Then must $\angle ABC$ be greater than $\angle ACB$.*

From $AC$ cut off $AD=AB$, and join $BD$. I. 3.

Then $\because AB=AD$,

$\therefore \angle ADB = \angle ABD$, I. A.

And $\because CD$, a side of $\triangle BDC$, is produced to $A$.

$\therefore \angle ADB$ is greater than $\angle ACB$; I. 16

$\therefore$ also $\angle ABD$ is greater than $\angle ACB$.

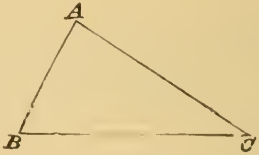
Much more is $\angle ABC$ greater than $\angle ACB$.

Q. E. D.

Ex. Shew that if two angles of a triangle be equal, the sides which subtend them are equal also (Eucl. I. 6).

### Proposition XIX. Theorem.

*If one angle of a triangle be greater than a second, the side opposite the first must be greater than that opposite the second.*



In $\triangle ABC$, let $\angle ABC$ be greater than $\angle ACB$.

*Then must $AC$ be greater than $AB$.*

For if $AC$ be not greater than $AB$,

$AC$ must either $= AB$, or be less than $AB$.

Now $AC$ cannot $= AB$, for then I. A.

$\angle ABC$ would $= \angle ACB$, which is not the case.

And $AC$ cannot be less than $AB$, for then I. 18.

$\angle ABC$ would be less than $\angle ACB$, which is not the case;

$\therefore$ $AC$ is greater than $AB$.

Q. E. D.

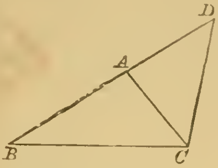
Ex. 1. In an obtuse-angled triangle, the greatest side is opposite the obtuse angle.

Ex. 2. $BC$, the base of an isosceles triangle $BAC$, is produced to any point $D$; shew that $AD$ is greater than $AB$.

Ex. 3. The perpendicular is the shortest straight line, which can be drawn from a given point to a given straight line; and of others, that which is nearer to the perpendicular is less than one more remote.

## PROPOSITION XX. THEOREM.

*Any two sides of a triangle are together greater than the third side.*



Let $ABC$ be a $\triangle$.

*Then any two of its sides must be together greater than the third side.*

Produce $BA$ to $D$, making $AD=AC$, and join $DC$.

Then $\because AD=AC$,

$\therefore \angle ACD = \angle ADC$, that is, $\angle BDC$. I. A.

Now $\angle BCD$ is greater than $\angle ACD$,

$\therefore \angle BCD$ is also greater than $\angle BDC$;

$\therefore BD$ is greater than $BC$. I. 19.

But $BD=BA$ and $AD$ together;

that is, $BD=BA$ and $AC$ together;

$\therefore BA$ and $AC$ together are greater than $BC$.

Similarly it may be shewn that

$AB$ and $BC$ together are greater than $AC$,

and $BC$ and $CA$ .............................. $AB$.

Q. E. D.

Ex. 1. Prove that any three sides of a quadrilateral figure are together greater than the fourth side.

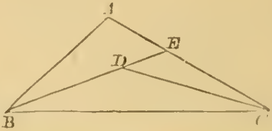
Ex. 2. Shew that any side of a triangle is greater than the difference between the other two sides.

Ex. 3. Prove that the sum of the distances of any point from the angular points of a quadrilateral is greater than half the perimeter of the quadrilateral.

Ex. 4. If one side of a triangle be bisected, the sum of the two other sides shall be more than double of the line joining the vertex and the point of bisection.

## Proposition XXI. Theorem.

*If, from the ends of the side of a triangle, there be drawn two straight lines to a point within the triangle; these will be together less than the other sides of the triangle, but will contain a greater angle.*



Let $ABC$ be a △, and from $D$, a pt. in the △, draw st. lines to $B$ and $C$.

*Then will $BD$, $DC$ together be less than $BA$, $AC$,*
*but ∠ $BDC$ will be greater than ∠ $BAC$.*

Produce $BD$ to meet $AC$ in $E$.

Then $BA$, $AE$ are together greater than $BE$. I. 20.

Add to each $EC$.

Then $BA$, $AC$ are together greater than $BE$, $EC$.

Again, $DE$, $EC$ are together greater than $DC$. I. 20.

Add to each $BD$.

Then $BE$, $EC$ are together greater than $BD$, $DC$.

And it has been shewn that $BA$, $AC$ are together greater than $BE$, $EC$;

∴ $BA$, $AC$ are together greater than $BD$, $DC$.

Next, ∵ ∠ $BDC$ is greater than ∠ $DEC$, I. 16.

and ∠ $DEC$ is greater than ∠ $BAC$, I. 16.

∴ ∠ $BDC$ is greater than ∠ $BAC$.

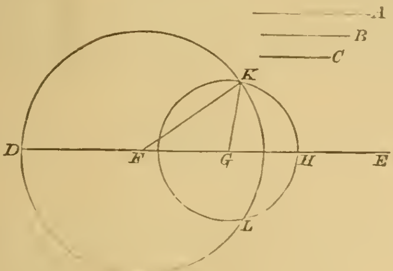
Q. E. D.

Ex. 1. Upon the base $AB$ of a triangle $ABC$ is described a quadrilateral figure $ADEB$, which is entirely within the triangle. Shew that the sides $AC$, $CB$ of the triangle are together greater than the sides $AD$, $DE$, $EB$ of the quadrilateral.

Ex. 2. Shew that the sum of the straight lines, joining the angles of a triangle with a point within the triangle, is less than the perimeter of the triangle, and greater than half the perimeter.

### PROPOSITION XXII. PROBLEM.

*To make a triangle, of which the sides shall be equal to three given straight lines, any two of which are together greater than the third.*



Let $A$, $B$, $C$ be the three given lines, any two of which are together greater than the third.

*It is required to make a* △ *having its sides* = $A$, $B$, $C$ *respectively.*

Take a st. line $DE$ of unlimited length.

In $DE$ make $DF = A$, $FG = B$, and $GH = C$. I. 3.

With centre $F$ and distance $FD$, describe ⊙ $DKL$.

With centre $G$ and distance $GH$, describe ⊙ $HKL$.

Join $FK$ and $GK$.

Then △ $KFG$ has its sides $= A$, $B$, $C$ respectively.

For $FK = FD$; Def. 13.

∴ $FK = A$:

and $GK = GH$; Def. 13.

∴ $GK = C$;

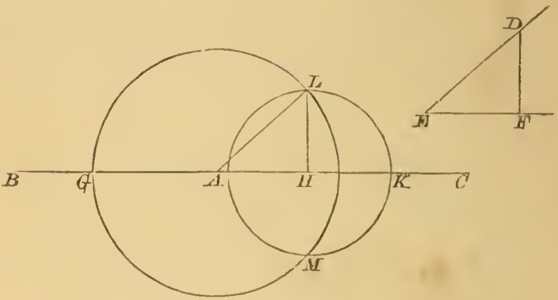
and $FG = B$;

∴ a △ $KFG$ has been described as reqd. Q. E. F.

Ex. Draw an isosceles triangle having each of the equal sides double of the base.

### PROPOSITION XXIII. PROBLEM.

*At a given point in a given straight line, to make an angle equal to a given angle.*



Let $A$ be the given pt., $BC$ the given line, $DEF$ the given $\angle$.

*It is reqd. to make at pt. $A$ an angle $= \angle DEF$.*

In $ED$, $EF$ take any pts. $D$, $F$; and join $DF$.

In $AB$, produced if necessary, make $AG = DE$.

In $AC$, produced if necessary, make $AH = EF$.

In $HC$, produced if necessary, make $HK = FD$.

With centre $A$, and distance $AG$, describe ⊙ $GLM$.

With centre $H$, and distance $HK$, describe ⊙ $LKM$.

Join $AL$ and $HL$.

Then $\because LA = AG$, $\therefore LA = DE$; Ax. 1

and $\because HL = HK$, $\therefore HL = FD$. Ax. 1

Then in $\triangle$s $LAH$, $DEF$,

$\because LA = DE$, and $AH = EF$, and $HL = FD$;

$\therefore \angle LAH = \angle DEF$. I. 8

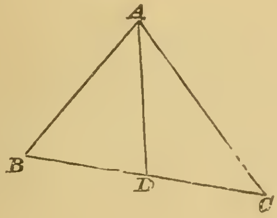
$\therefore$ an angle $LAH$ has been made at pt. $A$ as was reqd.

Q. E. F.

NOTE.—We here give the proof of a theorem, necessary to the proof of Prop. XXIV. and applicable to several propositions in Book III.

### PROPOSITION D. THEOREM.

*Every straight line, drawn from the vertex of a triangle to the base, is less than the greater of the two sides, or than either, if they be equal.*



In the △ $ABC$, let the side $AC$ be not less than $AB$.

Take any pt. $D$ in $BC$, and join $AD$.

*Then must $AD$ be less than $AC$.*

For ∵ $AC$ is not less than $AB$;

∴ ∠ $ABD$ is not less than ∠ $ACD$. I. A. and 18.

But ∠ $ADC$ is greater than ∠ $ABD$; I. 16.

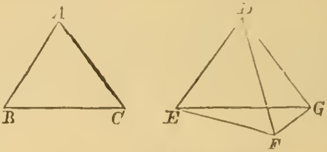
∴ ∠ $ADC$ is greater than ∠ $ACD$;

∴ $AC$ is greater than $AD$. I. 19.

Q. E. D.

### PROPOSITION XXIV. THEOREM.

*If two triangles have two sides of the one equal to two sides of the other, each to each, but the angle contained by the two sides of one of them greater than the angle contained by the two sides equal to them of the other; the base of that which has the greater angle must be greater than the base of the other.*



In the $\triangle$s $ABC$, $DEF$,
let $AB = DE$ and $AC = DF$,
and let $\angle BAC$ be greater than $\angle EDF$.
*Then must $BC$ be greater than $EF$.*

Of the two sides $DE$, $DF$ let $DE$ be not greater than $DF$.*
At pt. $D$ in st. line $ED$ make $\angle EDG = \angle BAC$, I. 23.
and make $DG = AC$ or $DF$, and join $EG$, $GF$.
Then $\because AB = DE$, and $AC = DG$, and $\angle BAC = \angle EDG$,
$\therefore BC = EG$, I. 4.
Again, $\because DG = DF$,
$\therefore \angle DFG = \angle DGF$; I. A.
$\therefore \angle EFG$ is greater than $\angle DGF$;
much more then $\angle EFG$ is greater than $\angle EGF$;
$\therefore EG$ is greater than $EF$. I. 19.
But $EG = BC$;
$\therefore BC$ is greater than $EF$.

Q. E. D.

* This line was added by Simson to obviate a defect in Euclid's proof. *Without* this condition, three distinct cases must be discussed. *With* the condition, we can prove that $F$ must lie below $EG$.

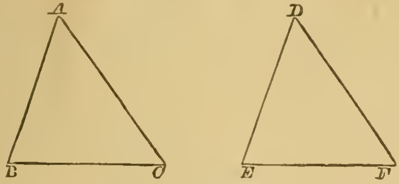
For since $DF$ is not less than $DE$, and $DG$ is drawn equal to $DF$, $DG$ is not less than $DE$.

Hence by Prop. D, any line drawn from $D$ to meet $EG$ is less than $DG$, and therefore $DF$, being equal to $DG$, must extend beyond $EG$.

For another method of proving the Proposition, see p. 113.

## PROPOSITION XXV. THEOREM.

*If two triangles have two sides of the one equal to two sides of the other, each to each, but the base of the one greater than the base of the other; the angle also, contained by the sides of that which has the greater base, must be greater than the angle contained by the sides equal to them of the other.*



In the $\triangle$s $ABC$, $DEF$,
let $AB = DE$ and $AC = DF$,
and let $BC$ be greater than $EF$.

*Then must* $\angle BAC$ *be greater than* $\angle EDF$.

For $\angle BAC$ is greater than, equal to, or less than $\angle EDF$.

Now $\angle BAC$ cannot $= \angle EDF$,

for then, by I. 4, $BC$ would $= EF$; which is not the case.

And $\angle BAC$ cannot be less than $\angle EDF$,

for then, by I. 24, $BC$ would be less than $EF$; which is not the case;

$\therefore \angle BAC$ must be greater than $\angle EDF$.

Q. E. D.

NOTE.—In Prop. XXVI. Euclid includes two cases, in which two triangles are equal in all respects; viz., when the following parts are equal in the two triangles:

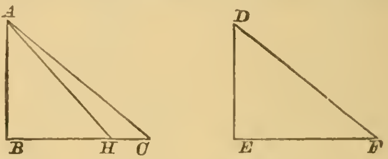
1. Two angles and the side between them.
2. Two angles and the side opposite one of them.

Of these we have already proved the first case, in Prop. B, so that we have only the second case left, to form the subject of Prop. XXVI., which we shall prove by the method of superposition.

For Euclid's proof of Prop. XXVI., see pp. 114-115.

## PROPOSITION XXVI. THEOREM.

*If two triangles have two angles of the one equal to two angles of the other, each to each, and one side equal to one side, those sides being opposite to equal angles in each; then must the triangles be equal in all respects.*



In $\triangle$s $ABC$, $DEF$,

let $\angle ABC = \angle DEF$, and $\angle ACB = \angle DFE$, and $AB = DE$.

*Then must* $BC = EF$, *and* $AC = DF$, *and* $\angle BAC = \angle EDF$.

Suppose $\triangle DEF$ to be applied to $\triangle ABC$,

so that $D$ coincides with $A$, and $DE$ falls on $AB$.

Then $\because DE = AB$, $\therefore E$ will coincide with $B$;

and $\because \angle DEF = \angle ABC$, $\therefore EF$ will fall on $BC$.

Then must $F$ coincide with $C$: for, if not,

let $F$ fall *between* $B$ and $C$, at the pt. $H$. Join $AH$.

Then $\because \angle AHB = \angle DFE$, I. 4.

$\therefore \angle AHB = \angle ACB$,

the extr. $\angle$ = the intr. and opposite $\angle$, which is impossible.

$\therefore F$ does not fall between $B$ and $C$.

Similarly, it may be shewn that $F$ does not fall on $BC$ produced.

$\therefore F$ coincides with $C$, and $\therefore BC = EF$;

$\therefore AC = DF$, and $\angle BAC = \angle EDF$, I. 4

and $\therefore$ the triangles are equal in all respects.

Q. E. D.

### *Miscellaneous Exercises on Props. I. to XXVI.*

1. $M$ is the middle point of the base $BC$ of an isosceles triangle $ABC$, and $N$ is a point in $AC$. Shew that the difference between $MB$ and $MN$ is less than that between $AB$ and $AN$.

2. $ABC$ is a triangle, and the angle at $A$ is bisected by a straight line which meets $BC$ at $D$; shew that $BA$ is greater than $BD$, and $CA$ greater than $CD$.

3. $AB$, $AC$ are straight lines meeting in $A$, and $D$ is a given point. Draw through $D$ a straight line cutting off equal parts from $AB$, $AC$.

4. Draw a straight line through a given point, to make equal angles with two given straight lines which meet.

5. A given angle $BAC$ is bisected; if $CA$ be produced to $G$ and the angle $BAG$ bisected, the two bisecting lines are at right angles.

6. Two straight lines are drawn to the base of a triangle from the vertex, one bisecting the vertical angle, and the other bisecting the base. Prove that the latter is the greater of the two lines.

7. Shew that Prop. XVII. may be proved without producing a side of the triangle.

8. Shew that Prop. XVIII. may be proved by means of the following construction: cut off $AD=AB$, draw $AE$, bisecting $\angle BAC$ and meeting $BC$ in $E$, and join $DE$.

9. Shew that Prop. XX. can be proved, without producing one of the sides of the triangle, by bisecting one of the angles.

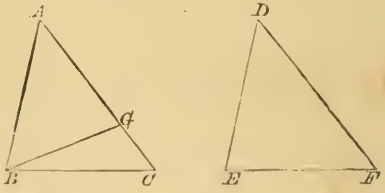
10. Given two angles of a triangle and the side adjacent to them, construct the triangle.

11. Shew that the perpendiculars, let fall on two sides of a triangle from any point in the straight line bisecting the angle contained by the two sides, are equal.

We conclude Section I. with the proof (omitted by Euclid) of another case in which two triangles are equal in all respects.

### PROPOSITION E. THEOREM.

*If two triangles have one angle of the one equal to one angle of the other, and the sides about a second angle in each equal: then, if the third angles in each be both acute, both obtuse, or if one of them be a right angle, the triangles are equal in all respects.*



In the $\triangle$s $ABC$, $DEF$, let $\angle BAC = \angle EDF$, $AB = DE$, $BC = EF$, and let $\angle$s $ACB$, $DFE$ be both acute, both obtuse, or let one of them be a right angle.

*Then must* $\triangle$*s* $ABC$, $DEF$ *be equal in all respects.*

For if $AC$ be not $= DF$, make $AG = DF$; and join $BG$.

Then in $\triangle$s $BAG$, $EDF$,

$\because BA = ED$, and $AG = DF$, and $\angle BAG = \angle EDF$,

$\therefore BG = EF$ and $\angle AGB = \angle DFE$. I. 4.

But $BC = EF$, and $\therefore BG = BC$;

$\therefore \angle BCG = \angle BGC$. I. A.

First, let $\angle ACB$ and $\angle DFE$ be both acute,

then $\angle AGB$ is acute, and $\therefore \angle BGC$ is obtuse; I. 13.

$\therefore \angle BCG$ is obtuse, which is contrary to the hypothesis.

Next, let $\angle ACB$ and $\angle DFE$ be both obtuse,

then $\angle AGB$ is obtuse, and $\therefore \angle BGC$ is acute; I. 13.

$\therefore \angle BCG$ is acute, which is contrary to the hypothesis.

Lastly, let one of the third angles $ACB$, $DFE$ be right angle.

If $\angle ACB$ be a rt. $\angle$,

then $\angle BGC$ is also a rt. $\angle$; I. A

$\therefore$ $\angle$s $BCG$, $BGC$ together = two rt. $\angle$s, which is impossible. I. 17.

Again, if $\angle DFE$ be a rt. $\angle$,

then $\angle AGB$ is a rt. $\angle$, and $\therefore$ $\angle BGC$ is a rt. $\angle$. I. 13.

Hence $\angle BCG$ is also a rt. $\angle$.

$\therefore$ $\angle$s $BCG$, $BGC$ together = two rt. $\angle$s, which is impossible. I. 17.

Hence $AC$ is equal to $DF$,

and the $\triangle$s $ABC$, $DEF$ are equal in all respects.

Q. E. D.

Cor. From the first case of this proposition we deduce the following important theorem:

*If two right-angled triangles have the hypotenuse and one side of the one equal respectively to the hypotenuse and one side of the other, the triangles are equal in all respects.*

Note. In the enunciation of Prop. E, if, instead of the words *if one of them be a right angle*, we put the words *both right angles*, this case of the proposition would be identical with I. 26.

## SECTION II.

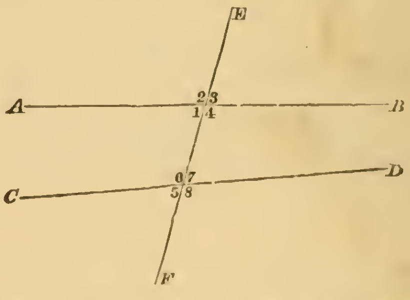
### *The Theory of Parallel Lines.*

### INTRODUCTION.

We have detached the Propositions, in which Euclid treats of Parallel Lines, from those which precede and follow them in the First Book, in order that the student may have a clearer notion of the difficulties attending this division of the subject, and of the way in which Euclid proposes to meet them.

We must first explain some technical terms used in this Section.

If a straight line *EF* cut two other straight lines *AB*, *CD*, it makes with those lines eight angles, to which particular names are given.



The angles numbered 1, 4, 6, 7 are called *Interior* angles
.......................... 2, 3, 5, 8 ............ *Exterior* .........

The angles marked 1 and 7 are called *alternate* angles.

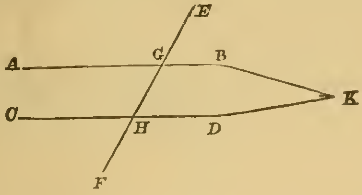
The angles marked 4 and 6 are also called alternate angles.

The pairs of angles 1 and 5, 2 and 6, 4 and 8, 3 and 7 are called *corresponding* angles.

Note. From I. 13 it is clear that the angles 1, 4, 6, 7 are together equal to four right angles.

## PROPOSITION XXVII. THEOREM.

*If a straight line, falling upon two other straight lines, make the alternate angles equal to one another; these two straight lines must be parallel.*



Let the st. line $EF$, falling on the st. lines $AB$, $CD$,

make the alternate $\angle$s $AGH$, $GHD$ equal.

*Then must* $AB$ *be* || *to* $CD$.

For if not, $AB$ and $CD$ will meet, if produced, either towards $B$, $D$, or towards $A$, $C$.

Let them be produced and meet towards $B$, $D$ in $K$.

Then $GHK$ is a $\triangle$;

and $\therefore \angle AGH$ is greater than $\angle GHD$. I. 16.

But $\angle AGH = \angle GHD$, Hyp.

which is impossible.

$\therefore AB$, $CD$ do not meet when produced towards $B$, $D$.

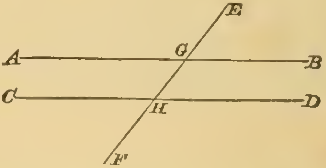
In like manner it may be shewn that they do not meet when produced towards $A$. $C$.

$AB$ and $CD$ are parallel. Def. 26.

Q. E. D.

## PROPOSITION XXVIII. THEOREM.

*If a straight line, falling upon two other straight lines, make the exterior angle equal to the interior and opposite upon the same side of the line, or make the interior angles upon the same side together equal to two right angles; the two straight lines are parallel to one another.*



Let the st. line $EF$, falling on st. lines $AB$, $CD$, make

I. $\angle EGB =$ corresponding $\angle GHD$, or

II. $\angle$s $BGH$, $GHD$ together $=$ two rt. $\angle$s.

*Then, in either case, $AB$ must be $\parallel$ to $CD$.*

I. $\because \angle EGB$ is given $= \angle GHD$, Hyp.

and $\angle EGB$ is known to be $= \angle AGH$, I. 15.

$\therefore \angle AGH = \angle GHD$;

and these are alternate $\angle$s;

$\therefore AB$ is $\parallel$ to $CD$. I. 27.

II. $\because \angle$s $BGH$, $GHD$ together $=$ two rt. $\angle$s, Hyp.

and $\angle$s $BGH$, $AGH$ together $=$ two rt. $\angle$s, I. 13.

$\therefore \angle$s $BGH$, $AGH$ together $= \angle$s $BGH$, $GHD$ together;

$\therefore \angle AGH = \angle GHD$;

$\therefore AB$ is $\parallel$ to $CD$. I. 27.

Q. E. D.

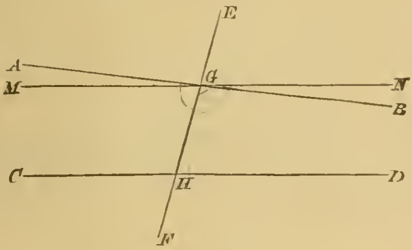
### Note 5. *On the Sixth Postulate.*

In the place of Euclid's Sixth Postulate many modern writers on Geometry propose, as more evident to the senses, the following Postulate :—

"*Two straight lines which cut one another cannot* BOTH *be parallel to the same straight line.*"

If this be assumed, we can prove Post. 6, as a Theorem, thus :

Let the line *EF* falling on the lines *AB*, *CD* make the $\angle$ s *BGH*, *GHD* together less than two rt. $\angle$ s. Then must *AB*, *CD* meet when produced towards *B*, *D*.



For if not, suppose *AB* and *CD* to be parallel.

Then $\because$ $\angle$ s *AGH*, *BGH* together = two rt. $\angle$ s, I. 13.

and $\angle$ s *GHD*, *BGH* are together less than two rt. $\angle$ s,

$\therefore$ $\angle$ *AGH* is greater than $\angle$ *GHD*.

Make $\angle$ *MGH* = $\angle$ *GHD*, and produce *MG* to *N*.

Then $\because$ the alternate $\angle$ s *MGH*, *GHD* are equal,

$\therefore$ *MN* is ∥ to *CD*. I. 27.

Thus two lines *MN*, *AB* which cut one another are both parallel to *CD*, which is impossible.

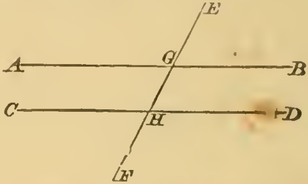
$\therefore$ *AB* and *CD* are not parallel.

It is also clear that they meet towards *B*, *D*, because *GB* lies between *GN* and *HD*.

Q. E. D.

### PROPOSITION XXIX. THEOREM.

*If a straight line fall upon two parallel straight lines, it makes the two interior angles upon the same side together equal to two right angles, and also the alternate angles equal to one another, and also the exterior angle equal to the interior and opposite upon the same side.*



Let the st. line $EF$ fall on the parallel st. lines $AB$, $CD$.

*Then must*

I. $\angle$s $BGH$, $GHD$ together $=$ two rt. $\angle$s.
II. $\angle AGH =$ alternate $\angle GHD$.
III. $\angle EGB =$ corresponding $\angle GHD$.

I. $\angle$s $BGH$, $GHD$ cannot be together *less* than two rt. $\angle$s,
for then $AB$ and $CD$ would meet if produced towards $B$ and $D$, Post. 6.
which cannot be, for they are parallel.
Nor can $\angle$s $BGH$, $GHD$ be together *greater* than two rt. $\angle$s,
for then $\angle$s $AGH$, $GHC$ would be together less than two rt. $\angle$s, I. 13.
and $AB$, $CD$ would meet if produced towards $A$ and $C$ Post. 6
which cannot be, for they are parallel,
∴ $\angle$s $BGH$, $GHD$ together $=$ two rt. $\angle$s.

II. ∵ $\angle$s $BGH$, $GHD$ together $=$ two rt. $\angle$s,
and $\angle$s $BGH$, $AGH$ together $=$ two rt. $\angle$s, I. 13.
∴ $\angle$s $BGH$, $AGH$ together $= \angle$s $BGH$, $GHD$ together,
and ∴ $\angle AGH = \angle GHD$. Ax. 3.

III. ∵ $\angle AGH = \angle GHD$,
and $\angle AGH = \angle EGB$, I. 15.
∴ $\angle EGB = \angle GHD$. Ax. 1

Q. E. D.

### EXERCISES.

1. If through a point, equidistant from two parallel straight lines, two straight lines be drawn cutting the parallel straight lines; they will intercept equal portions of the parallel lines.

2. If a straight line be drawn, bisecting one of the angles of a triangle, to meet the opposite side; the straight lines drawn from the point of section, parallel to the other sides and terminated by those sides, will be equal.

3. If any straight line joining two parallel straight lines be bisected, any other straight line, drawn through the point of bisection to meet the two lines, will be bisected in that point.

NOTE. One Theorem (A) is said to be the *converse* of another Theorem (B), when the hypothesis in (A) is the conclusion in (B), and the conclusion in (A) is the hypothesis in (B).

For example, the Theorem I. A. may be stated thus:

*Hypothesis.* If two sides of a triangle be equal.

*Conclusion.* The angles opposite those sides must also be equal.

The converse of this is the Theorem I. B. Cor.:

*Hypothesis.* If two angles of a triangle be equal.

*Conclusion.* The sides opposite those angles must also be equal.

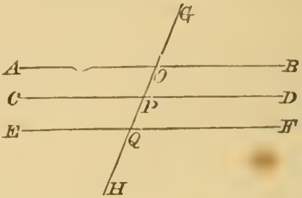
The following are other instances:

Postulate VI. is the converse of I. 17.

I. 29 is the converse of I. 27 and 28.

### PROPOSITION XXX. THEOREM.

*Straight lines which are parallel to the same straight line are parallel to one another.*



Let the st. lines *AB*, *CD* be each ‖ to *EF*.

*Then must AB be ‖ to CD.*

Draw the st. line *GH*, cutting *AB*, *CD*, *EF* in the pts. *O*, *P*, *Q*.

Then ∵ *GH* cuts the ‖ lines *AB*, *EF*,

∴ $\angle AOP$=alternate $\angle PQF$. I. 29.

And ∵ *GH* cuts the ‖ lines *CD*, *EF*,

∴ extr. $\angle OPD$=intr. $\angle PQF$, I. 29.

∴ $\angle AOP = \angle OPD$;

and these are alternate angles;

∴ *AB* is ‖ to *CD* I. 27.

Q. E. D.

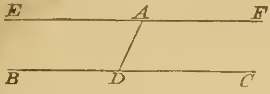
The following Theorems are important. They admit of easy proof, and are therefore left as Exercises for the student.

1. If two straight lines be parallel to two other straight lines, each to each, the first pair make the same angles with one another as the second.

2. If two straight lines be perpendicular to two other straight lines, each to each, the first pair make the same angles with one another as the second.

## PROPOSITION XXXI. PROBLEM.

*To draw a straight line through a given point parallel to a given straight line.*



Let $A$ be the given pt. and $BC$ the given st. line.

*It is required to draw through $A$ a st. line $\parallel$ to $BC$.*

In $BC$ take any pt. $D$, and join $AD$.

Make $\angle DAE = \angle ADC$. I. 23.

Produce $EA$ to $F$. Then $EF$ shall be $\parallel$ to $BC$.

For $\because$ $AD$, meeting $EF$ and $BC$, makes the alternate angles equal, that is, $\angle EAD = \angle ADC$,

$\therefore$ $EF$ is $\parallel$ to $BC$. I. 27.

$\therefore$ a st. line has been drawn through $A$ $\parallel$ to $BC$.

Q. E. F.

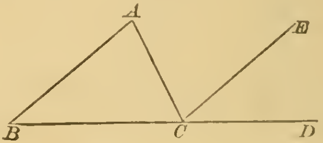
Ex. 1. From a given point draw a straight line, to make an angle with a given straight line that shall be equal to a given angle.

Ex. 2. Through a given point $A$ draw a straight line $ABC$, meeting two parallel straight lines in $B$ and $C$, so that $BC$ may be equal to a given straight line.

3 ✓

### PROPOSITION XXXII. THEOREM.

*If a side of any triangle be produced, the exterior angle is equal to the two interior and opposite angles, and the three interior angles of every triangle are together equal to two right angles.*



Let $ABC$ be a $\triangle$, and let one of its sides, $BC$, be produced to $D$.

*Then will*

I. $\angle ACD = \angle$s $ABC$, $BAC$ *together*.

II. $\angle$s $ABC$, $BAC$, $ACB$ *together = two rt.* $\angle$*s.*

From $C$ draw $CE \parallel$ to $AB$. I. 31.

Then I. $\because$ $BD$ meets the $\parallel$s $EC$, $AB$,

$\therefore$ extr. $\angle ECD =$ intr. $\angle ABC$. I. 29.

And $\because$ $AC$ meets the $\parallel$s $EC$, $AB$,

$\therefore$ $\angle ACE =$ alternate $\angle BAC$. I. 29.

.. $\angle$s $ECD$, $ACE$ together $= \angle$s $ABC$, $BAC$ together ;

$\therefore$ $\angle ACD = \angle$s $ABC$, $BAC$ together.

And II. $\because$ $\angle$s $ABC$, $BAC$ together $= \angle ACD$,

to each of these equals add $\angle ACB$ ;

then $\angle$s $ABC$, $BAC$, $ACB$ together $= \angle$s $ACD$, $ACB$ together,

$\therefore$ $\angle$s $ABC$, $BAC$, $ACB$ together = two rt. $\angle$s. I. 13.

Q. E. D.

Ex. 1. In an acute-angled triangle, any two angles are greater than the third.

Ex. 2. The straight line, which bisects the external vertical angle of an isosceles triangle is parallel to the base.

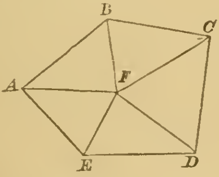
Ex. 3 If the side $BC$ of the triangle $ABC$ be produced to $D$, and $AE$ be drawn bisecting the angle $BAC$ and meeting $BC$ in $E$; shew that the angles $ABD$, $ACD$ are together double of the angle $AED$.

Ex. 4. If the straight lines bisecting the angles at the base of an isosceles triangle be produced to meet; shew that they will contain an angle equal to an exterior angle at the base of the triangle.

Ex. 5. If the straight line bisecting the external angle of a triangle be parallel to the base; prove that the triangle is isosceles.

The following Corollaries to Prop. 32 were first given in Simson's Edition of Euclid.

Cor. 1. *The sum of the interior angles of any rectilinear figure together with four right angles is equal to twice as many right angles as the figure has sides.*



Let $ABCDE$ be any rectilinear figure.

Take any pt. $F$ within the figure, and from $F$ draw the st. lines $FA$, $FB$, $FC$, $FD$, $FE$ to the angular pts. of the figure

Then there are formed as many $\triangle$s as the figure has sides.

The three $\angle$s in *each* of these $\triangle$s together $=$ two rt. $\angle$s.

$\therefore$ *all* the $\angle$s in these $\triangle$s together $=$ twice as many right $\angle$s as there are $\triangle$s, that is, twice as many right $\angle$s as the figure has sides.

Now angles of all the $\triangle$s $=$ $\angle$s at $A$, $B$, $C$, $D$, $E$ and $\angle$s at $F$,

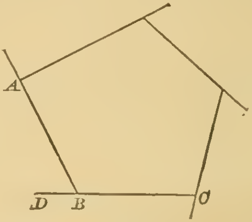
that is, $=$ $\angle$s of the figure and $\angle$s at $F$,

and $\therefore$ $=$ $\angle$s of the figure and four rt. $\angle$s. I. 15. Cor 2

$\therefore$ $\angle$s of the figure and four rt. $\angle$s $=$ twice as many rt. $\angle$s as the figure has sides.

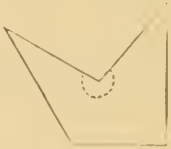
Cor. 2. *The exterior angles of any convex rectilinear figure, made by producing each of its sides in succession, are together equal to four right angles.*

Every interior angle, as *ABC*, and its adjacent exterior angle, as *ABD*, together are = two rt. ∠ s.



∴ all the intr. ∠ s together with all the extr. ∠ s
= twice as many rt. ∠ s as the figure has sides.
But all the intr. ∠ s together with four rt. ∠ s
= twice as many rt. ∠ s as the figure has sides.
∴ all the intr. ∠ s together with all the extr. ∠ s
= all the intr. ∠ s together with four rt. ∠ s.
∴ all the extr. ∠ s = four rt. ∠ s.

Note. The latter of these corollaries refers only to *convex* figures, that is, figures in which every interior angle is less than two right angles. When a figure contains an angle greater



than two right angles, as the angle marked by the dotted line in the diagram, this is called a reflex *angle*. See p. 149.

Ex. 1. The exterior angles of a quadrilateral made by producing the sides successively are together equal to the interior angles.

Ex. 2. Prove that the interior angles of a hexagon are equal to eight right angles.

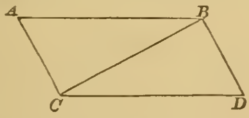
Ex. 3. Shew that the angle of an equiangular pentagon is $\frac{6}{5}$ of a right angle.

Ex. 4. How many sides has the rectilinear figure, the sum of whose interior angles is double that of its exterior angles?

Ex. 5. How many sides has an equiangular polygon, four of whose angles are together equal to seven right angles?

## PROPOSITION XXXIII. THEOREM.

*The straight lines which join the extremities of two equal and parallel straight lines, towards the same parts, are also themselves equal and parallel.*



Let the equal and ∥ st. lines $AB$, $CD$ be joined towards the same parts by the st. lines $AC$, $BD$.

*Then must $AC$ and $BD$ be equal and ∥.*

Join $BC$.

Then $\because AB$ is ∥ to $CD$,

$\therefore \angle ABC =$ alternate $\angle DCB$. I. 29.

Then in △s $ABC$, $BCD$,

$\because AB = CD$, and $BC$ is common, and $\angle ABC = \angle DCB$,

$\therefore AC = BD$, and $\angle ACB = \angle DBC$. I. 4.

Then $\because BC$, meeting $AC$ and $BD$,

makes the alternate ∠s $ACB$, $DBC$ equal,

$\therefore AC$ is ∥ to $BD$.

Q. E. D.

## *Miscellaneous Exercises on Sections I. and II.*

1. If two exterior angles of a triangle be bisected by straight lines which meet in $O$; prove that the perpendiculars from $O$ on the sides, or the sides produced, of the triangle are equal.

2. Trisect a right angle.

3. The bisectors of the three angles of a triangle meet in one point.

4. The perpendiculars to the three sides of a triangle drawn from the middle points of the sides meet in one point.

5. The angle between the bisector of the angle $BAC$ of the triangle $ABC$ and the perpendicular from $A$ on $BC$, is equal to half the difference between the angles at $B$ and $C$.

6. If the straight line $AD$ bisect the angle at $A$ of the triangle $ABC$, and $BDE$ be drawn perpendicular to $AD$, and meeting $AC$, or $AC$ produced, in $E$; shew that $BD$ is equal to $DE$.

7. Divide a right-angled triangle into two isosceles triangles.

8. $AB$, $CD$ are two given straight lines. Through a point $E$ between them draw a straight line $GEH$, such that the intercepted portion $GH$ shall be bisected in $E$.

9. The vertical angle $O$ of a triangle $OPQ$ is a right, acute, or obtuse angle, according as $OR$, the line bisecting $PQ$, is equal to, greater or less than the half of $PQ$.

10. Shew by means of Ex. 9 how to draw a perpendicular to a given straight line from its extremity without producing it.

## SECTION III.

### *On the Equality of Rectilinear Figures in respect of Area.*

THE amount of space enclosed by a Figure is called the Area of that figure.

Euclid calls two figures *equal* when they enclose the same amount of space. They may be dissimilar in shape, but if the areas contained within the boundaries of the figures be the same, then he calls the figures *equal.* He regards a triangle, for example, as a figure having sides and angles and area, and he proves in this section that two triangles may have equality of area, though the sides and angles of each may be unequal.

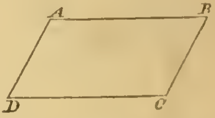
Coincidence of their boundaries is a test of the equality of all geometrical magnitudes, as we explained in Note 1, page 14.

In the case of lines and angles it is the only test: in the case of *figures* it is *a test, but not the only test;* as we shall shew in this Section.

The sign =, standing between the symbols denoting two *figures*, must be read *is equal in area to.*

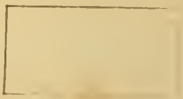
Before we proceed to prove the Propositions included in this Section, we must complete the list of Definitions required in Book I., continuing the numbers prefixed to the definitions in page 6.

## Definitions.



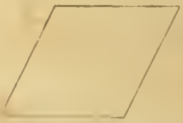
XXVII. A Parallelogram is a four-sided figure whose opposite sides are parallel.

For brevity we often designate a parallelogram by two letters only, which mark opposite angles. Thus we call the figure in the margin the parallelogram $AC$.



XXVIII. A Rectangle is a parallelogram, having one of its angles a right angle.

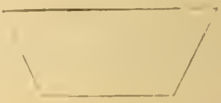
Hence by I. 29, *all* the angles of a rectangle are right angles.



XXIX. A Rhombus is a parallelogram, having its sides equal.

XXX. A Square is a parallelogram, having its sides equal and one of its angles a right angle.

Hence, by I. 29, *all* the angles of a square are right angles.



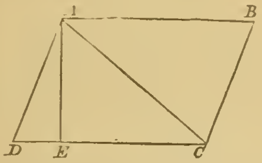
XXXI. A Trapezium is a four-sided figure of which two sides only are parallel.

XXXII. A Diagonal of a four-sided figure is the straight line joining two of the opposite angular points.

XXXIII. The Altitude of a Parallelogram is the perpendicular distance of one of its sides from the side opposite, regarded as the Base.

The altitude of a triangle is the perpendicular distance of one of its angular points from the side opposite, regarded as the base.

Thus if *ABCD* be a parallelogram, and *AE* a perpendicular let fall from *A* to *CD*. *AE* is the *altitude* of the parallelogram, and also of the triangle *ACD*.



If a perpendicular be let fall from *B* to *DC* produced, meeting *DC* in *F*, *BF* is the altitude of the parallelogram.

### Exercises.

Prove the following theorems :

1. The diagonals of a square make with each of the sides an angle equal to half a right angle.

2. If two straight lines bisect each other. the lines joining their extremities will form a parallelogram.

3. Straight lines bisecting two adjacent angles of a parallelogram intersect at right angles.

4. If the straight lines joining two opposite angular points of a parallelogram bisect the angles, the parallelogram has all its sides equal.

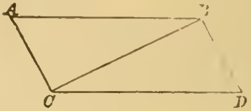
5. If the opposite angles of a quadrilateral be equal, the quadrilateral is a parallelogram.

6. If two opposite sides of a quadrilateral figure be equal to one another, and the two remaining sides be also equal to one another, the figure is a parallelogram.

7. If one angle of a rhombus be equal to two-thirds of two right angles, the diagonal drawn from that angular point divides the rhombus into two equilateral triangles.

## PROPOSITION XXXIV. THEOREM.

*The opposite sides and angles of a parallelogram are equal to one another, and the diagonal bisects it.*



Let $ABDC$ be a ▱, and $BC$ a diagonal of the ▱.

*Then must* $AB = DC$ and $AC = DB$,

*and* $\angle BAC = \angle CDB$, and $\angle ABD = \angle ACD$

*and* $\triangle ABC = \triangle DCB$.

For $\because AB$ is $\parallel$ to $CD$, and $BC$ meets them,

$\therefore \angle ABC =$ alternate $\angle DCB$, I. 29.

and $\because AC$ is $\parallel$ to $BD$, and $BC$ meets them,

$\therefore \angle ACB =$ alternate $\angle DBC$. I. 29.

Then in $\triangle$s $ABC$, $DCB$,

$\because \angle ABC = \angle DCB$, and $\angle ACB = \angle DBC$,

and $BC$ is common, a side adjacent to the equal $\angle$s in each;

$\therefore AB = DC$, and $AC = DB$, and $\angle BAC = \angle CDB$,

and $\triangle ABC = \triangle DCB$. I. B.

Also $\because \angle ABC = \angle DCB$, and $\angle DBC = \angle ACB$,

$\therefore \angle$s $ABC$, $DBC$ together $= \angle$s $DCB$, $ACB$ together,

that is, $\angle ABD = \angle ACD$.

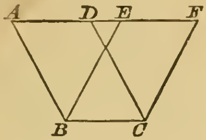
Q. E. D.

Ex. 1. Shew that the diagonals of a parallelogram bisect each other.

Ex. 2. Shew that the diagonals of a rectangle are equal.

## PROPOSITION XXXV. THEOREM.

*Parallelograms on the same base and between the same parallels are equal.*



Let the ▱s $ABCD$, $EBCF$ be on the same base $BC$ and between the same ∥s $AF$, $BC$.

*Then must* ▱ $ABCD =$ ▱ $EBCF$.

CASE I. If $AD$, $EF$ have no point common to both,

Then in the △s $FDC$, $EAB$,

∵ extr. $\angle FDC =$ intr. $\angle EAB$, I. 29.

and intr. $\angle DFC =$ extr. $\angle AEB$, I. 29.

and $DC = AB$, I. 34.

∴ △ $FDC =$ △ $EAB$. I. 26.

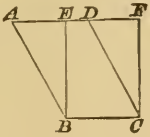
Now ▱ $ABCD$ with △ $FDC =$ figure $ABCF$;

and ▱ $EBCF$ with △ $EAB =$ figure $ABCF$;

∴ ▱ $ABCD$ with △ $FDC =$ ▱ $EBCF$ with △ $EAB$;

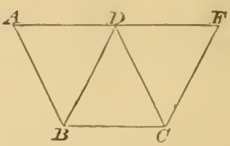
∴ ▱ $ABCD =$ ▱ $EBCF$.

CASE II. If the sides $AD$, $EF$ overlap one another



the same method of proof applies.

Case III. If the sides opposite to $BC$ be terminated in the same point $D$,



the same method of proof is applicable,
but it is easier to reason thus:

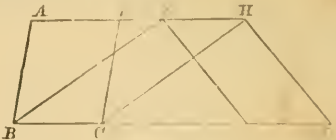
Each of the ▱s is double of △ $BDC$; I. 34.

$\therefore$ ▱ $ABCD$ = ▱ $DBCF$.

Q. E. D.

### Proposition XXXVI. Theorem.

*Parallelograms on equal bases, and between the same parallels, are equal to one another.*



Let the ▱s $ABCD$, $EFGH$ be on equal bases $BC$, $FG$, and between the same ‖s $AH$, $BG$.

*Then must* ▱ $ABCD$ = ▱ $EFGH$.

Join $BE$, $CH$.

Then $\because BC = FG$, Hyp.

and $EH = FG$; I. 34.

$\therefore BC = EH$;

and $BC$ is ‖ to $EH$. Hyp.

$\therefore EB$ is ‖ to $CH$; I. 33.

$\therefore EBCH$ is a parallelogram.

Now ▱ $EBCH$ = ▱ $ABCD$, I. 35.

$\because$ they are on the same base $BC$ and between the same ‖s;

and ▱ $EBCH$ = ▱ $EFGH$, I. 35.

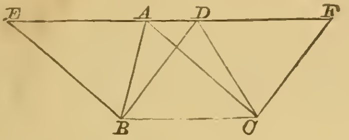
$\because$ they are on the same base $EH$ and between the same ‖s,

$\therefore$ ▱ $ABCD$ = ▱ $EFGH$.

Q. E. D.

## PROPOSITION XXXVII. THEOREM.

*Triangles upon the same base, and between the same parallels, are equal to one another.*



Let $\triangle$s $ABC$, $DBC$ be on the same base $BC$ and between the same $\parallel$s $AD$, $BC$.

*Then must* $\triangle\ ABC = \triangle\ DBC$.

From $B$ draw $BE \parallel$ to $CA$ to meet $DA$ produced in $E$.

From $C$ draw $CF \parallel$ to $BD$ to meet $AD$ produced in $F$.

Then $EBCA$ and $FCBD$ are parallelograms,

and $\square\ EBCA = \square\ FCBD$, I. 35.

$\because$ they are on the same base and between the same $\parallel$s.

Now $\triangle\ ABC$ is half of $\square\ EBCA$, I. 34.

and $\triangle\ DBC$ is half of $\square\ FCBD$; I. 34.

$\therefore\ \triangle\ ABC = \triangle\ DBC$. Ax. 7.

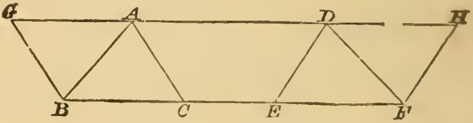
Q. E. D.

Ex. 1. If $P$ be a point in a side $AB$ of a parallelogram $ABCD$, and $PC$, $PD$ be joined, the triangles $PAD$, $PBC$ are together equal to the triangle $PDC$.

Ex. 2. If $A$, $B$ be points in one, and $C$, $D$ points in another of two parallel straight lines, and the lines $AD$, $BC$ intersect in $E$, then the triangles $AEC$, $BED$ are equal.

### Proposition XXXVIII. Theorem.

*Triangles upon equal bases, and between the same parallels, are equal to one another.*



Let $\triangle$s $ABC$, $DEF$ be on equal bases, $BC$, $EF$, and between the same ‖s $BF$, $AD$.

*Then must* $\triangle ABC = \triangle DEF$.

From $B$ draw $BG$ ‖ to $CA$ to meet $DA$ produced in $G$.

From $F$ draw $FH$ ‖ to $ED$ to meet $AD$ produced in $H$.

Then $CG$ and $EH$ are parallelograms, and they are equal,

$\because$ they are on equal bases $BC$, $EF$, and between the same ‖s $BF$, $GH$. I. 36

Now $\triangle ABC$ is half of ▱ $CG$,

and $\triangle DEF$ is half of ▱ $EH$;

$\therefore \triangle ABC = \triangle DEF$. Ax. 7.

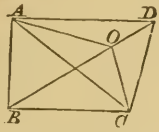
Q. E. D.

Ex. 1. Shew that a straight line, drawn from the vertex of a triangle to bisect the base, divides the triangle into two equal parts.

Ex. 2. In the equal sides $AB$, $AC$ of an isosceles triangle $ABC$ points $D$, $E$ are taken such that $BD = AE$. Shew that the triangles $CBD$, $ABE$ are equal.

### PROPOSITION XXXIX. THEOREM.

*Equal triangles upon the same base, and upon the same side of it, are between the same parallels.*



Let the equal $\triangle$s $ABC$, $DBC$ be on the same base $BC$, and on the same side of it.

Join $AD$.

*Then must $AD$ be $\parallel$ to $BC$.*

For if not, through $A$ draw $AO \parallel$ to $BC$, so as to meet $BD$, or $BD$ produced, in $O$, and join $OC$.

Then $\because$ $\triangle$s $ABC$, $OBC$ are on the same base and between the same $\parallel$s,

$\therefore \triangle ABC = \triangle OBC$. I. 37.

But $\triangle ABC = \triangle DBC$; Hyp.

$\therefore \triangle OBC = \triangle DBC$,

the less = the greater, which is impossible;

$\therefore AO$ is not $\parallel$ to $BC$.

In the same way it may be shewn that no other line passing through $A$ but $AD$ is $\parallel$ to $BC$;

$\therefore AD$ is $\parallel$ to $BC$.

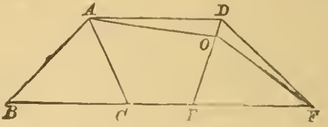
Q. E. D.

Ex. 1. $AD$ is parallel to $BC$; $AC$, $BD$ meet in $E$; $BC$ is produced to $P$ so that the triangle $PEB$ is equal to the triangle $ABC$: shew that $PD$ is parallel to $AC$.

Ex. 2. If of the four triangles into which the diagonals divide a quadrilateral, two opposite ones are equal, the quadrilateral has two opposite sides parallel.

### PROPOSITION XL. THEOREM.

*Equal triangles upon equal bases, in the same straight line, and towards the same parts, are between the same parallels.*



Let the equal △s $ABC$, $DEF$ be on equal bases $BC$, $EF$ in the same st. line $BF$ and towards the same parts.

Join $AD$.

*Then must $AD$ be ∥ to $BF$.*

For if not, through $A$ draw $AO$ ∥ to $BF$, so as to meet $ED$, or $ED$ produced, in $O$, and join $OF$.

Then △ $ABC$ = △ $OEF$, ∵ they are on equal bases and between the same ∥s. I. 38.

But △ $ABC$ = △ $DEF$; Hyp.

∴ △ $OEF$ = △ $DEF$,

the less = the greater, which is impossible.

∴ $AO$ is not ∥ to $BF$.

In the same way it may be shewn that no other line passing through $A$ but $AD$ is ∥ to $BF$,

∴ $AD$ is ∥ to $BF$.

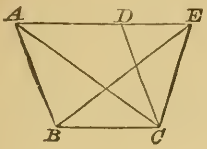
Q. E. D.

Ex. 1. The straight line, joining the points of bisection of two sides of a triangle, is parallel to the base, and is equal to half the base.

Ex. 2. The straight lines, joining the middle points of the sides of a triangle, divide it into four equal triangles.

### PROPOSITION XLI. THEOREM.

*If a parallelogram and a triangle be upon the same base, and between the same parallels, the parallelogram is double of the triangle.*



Let the ▱ *ABCD* and the △ *EBC* be on the same base *BC* and between the same ||s *AE*, *BC*.

*Then must ▱ ABCD be double of △ EBC.*

Join *AC*.

Then △ *ABC* = △ *EBC*, ∵ they are on the same base and between the same ||s ; I. 37.

and ▱ *ABCD* is double of △ *ABC*, ∵ *AC* is a diagonal of *ABCD* ; I. 34.

∴ ▱ *ABCD* is double of △ *EBC*.

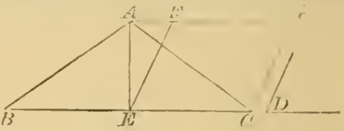
Q. E. D.

Ex. 1. If from a point, without a parallelogram, there be drawn two straight lines to the extremities of the two opposite sides, between which, when produced, the point does not lie, the difference of the triangles thus formed is equal to half the parallelogram.

Ex. 2. The two triangles, formed by drawing straight lines from any point within a parallelogram to the extremities of its opposite sides, are together half of the parallelogram.

### Proposition XLII. Problem.

*To describe a parallelogram that shall be equal to a given triangle, and have one of its angles equal to a given angle.*



Let $ABC$ be the given $\triangle$, and $D$ the given $\angle$.

*It is required to describe a ▱ equal to $\triangle\ ABC$, having one of its $\angle$ s $= \angle\ D$.*

Bisect $BC$ in $E$ and join $AE$. I. 10.

At $E$ make $\angle\ CEF = \angle\ D$. I. 23.

Draw $AFG$ ∥ to $BC$, and from $C$ draw $CG$ ∥ to $EF$.

Then $FECG$ is a parallelogram.

Now $\triangle\ AEB = \triangle\ AEC$,

∵ they are on equal bases and between the same ||s. I. 38.

∴ $\triangle\ ABC$ is double of $\triangle\ AEC$.

But ▱ $FECG$ is double of $\triangle\ AEC$,

∵ they are on same base and between same ||s. I. 41.

∴ ▱ $FECG = \triangle\ ABC$, Ax. 6.

and ▱ $FECG$ has one of its $\angle$ s, $CEF = \angle\ D$.

∴ ▱ $FECG$ has been described as was reqd.

Q. E. F.

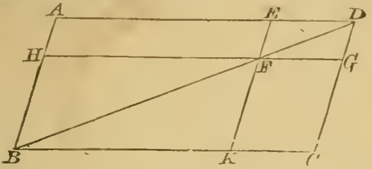
Ex. 1. Describe a triangle, which shall be equal to a given parallelogram, and have one of its angles equal to a given rectilineal angle.

Ex. 2. Construct a parallelogram, equal to a given triangle, and such that the sum of its sides shall be equal to the sum of the sides of the triangle.

Ex. 3. The perimeter of an isosceles triangle is greater than the perimeter of a rectangle, which is of the same altitude with, and equal to, the given triangle.

### Proposition XLIII. Theorem.

*The complements of the parallelograms, which are about the diameter of any parallelogram, are equal to one another.*



Let $ABCD$ be a ▱, of which $BD$ is a diagonal, and $EG$, $HK$ the ▱s about $BD$, that is, through which $BD$ passes,

and $AF$, $FC$ the other ▱s, which make up the whole figure $ABCD$,

and which are $\therefore$ called the Complements.

*Then must complement $AF$ = complement $FC$.*

For $\because$ $BD$ is a diagonal of ▱ $AC$,
$\therefore \triangle ABD = \triangle CDB$; I. 34.
and $\because$ $BF$ is a diagonal of ▱ $HK$,
$\therefore \triangle HBF = \triangle KFB$; I. 34.
and $\because$ $FD$ is a diagonal of ▱ $EG$,
$\therefore \triangle EFD = \triangle GDF$ I. 34.

Hence sum of $\triangle$s $HBF$, $EFD$ = sum of $\triangle$s $KFB$, $GDF$.

Take these equals from $\triangle$s $ABD$, $CDB$ respectively,
then remaining ▱ $AF$ = remaining ▱ $FC$. Ax. 3.

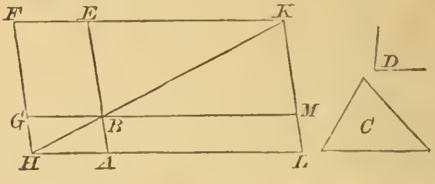
Q. E. D.

Ex. 1. If through a point $O$, within a parallelogram $ABCD$, two straight lines are drawn parallel to the sides, and the parallelograms $OB$, $OD$ are equal; the point $O$ is in the diagonal $AC$.

Ex. 2. $ABCD$ is a parallelogram, $AMN$ a straight line meeting the sides $BC$, $CD$ (one of them being produced) in $M$, $N$. Shew that the triangle $MBN$ is equal to the triangle $MDC$.

### PROPOSITION XLIV. PROBLEM.

*To a given straight line to apply a parallelogram, which shall be equal to a given triangle, and have one of its angles equal to a given angle.*



Let *AB* be the given st. line, *C* the given △, *D* the given ∠.

*It is required to apply to AB a ▱ = △ C and having one of its ∠ s = ∠ D.*

Make a ▱ = △ *C*, and having one of its angles = ∠ *D*, I. 42.

and suppose it to be removed to such a position that one of the sides containing this angle is in the same st. line with *AB*, and let the ▱ be denoted by *BEFG*.

Produce *FG* to *H*, draw *AH* ∥ to *BG* or *EF*, and join *BH*.

Then ∵ *FH* meets the ∥s *AH*, *EF*,

∴ sum of ∠ s *AHF*, *HFE* = two rt. ∠ s; I. 29

∴ sum of ∠ s *BHG*, *HFE* is less than two rt. ∠ s;

∴ *HB*, *FE* will meet if produced towards *B*, *E*. Post. 6

Let them meet in *K*.

Through *K* draw *KL* ∥ to *EA* or *FH*,

and produce *HA*, *GB* to meet *KL* in the pts. *L*, *M*.

Then *HFKL* is a ▱, and *HK* is its diagonal;

and *AG*, *ME* are ▱s about *HK*,

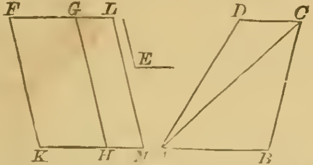
∴ complement *BL* = complement *BF*, I. 43

∴ ▱ *BL* = △ *C*.

Also the ▱ *BL* has one of its ∠ s, *ABM* = ∠ *EBG*, and ∴ equal to ∠ *D*.

### PROPOSITION XLV. PROBLEM.

*To describe a parallelogram, which shall be equal to a given rectilinear figure, and have one of its angles equal to a given angle.*



Let $ABCD$ be the given rectil. figure, and $E$ the given $\angle$.

*It is required to describe a* ▱ $=$ *to* $ABCD$, *having one of its* $\angle$*s* $= \angle E$.

Join $AC$.

Describe a ▱ $FGHK = \triangle ABC$, having $\angle FKH = \angle E$. I. 42.

To $GH$ apply a ▱ $GHML = \triangle CDA$, having $\angle GHM = \angle E$. I. 44.

Then $FKML$ is the ▱ reqd.

For ∵ $\angle GHM$ and $\angle FKH$ are each $= \angle E$;
∴ $\angle GHM = \angle FKH$,
∴ sum of $\angle$s $GHM$, $GHK$ = sum of $\angle$s $FKH$, $GHK$
= two rt. $\angle$s; I. 29.
∴ $KHM$ is a st. line. I. 14.

Again, ∵ $HG$ meets the ∥s $FG$, $KM$,
$\angle FGH = \angle GHM$,
∴ sum of $\angle$s $FGH$, $LGH$ = sum of $\angle$s $GHM$, $LGH$
= two rt. $\angle$s; I. 29.
∴ $FGL$ is a st. line. I. 14.

Then ∵ $KF$ is ∥ to $HG$, and $HG$ is ∥ to $LM$
∴ $KF$ is ∥ to $LM$; I. 30.
and $KM$ has been shewn to be ∥ to $FL$,
∴ $FKML$ is a parallelogram,
and ∵ $FH = \triangle ABC$, and $GM = \triangle CDA$,
∴ ▱ $FM$ = whole rectil. fig. $ABCD$,
and ▱ $FM$ has one of its $\angle$s, $FKM = \angle E$.

In the same way a ▱ may be constructed equal to a given rectil. fig. of any number of sides, and having one of its angles equal to a given angle. Q. E. F.

### *Miscellaneous Exercises.*

1. If one diagonal of a quadrilateral bisect the other, it divides the quadrilateral into two equal triangles.

2. If from any point in the diagonal, or the diagonal produced, of a parallelogram, straight lines be drawn to the opposite angles, they will cut off equal triangles.

3. In a trapezium the straight line, joining the middle points of the parallel sides, bisects the trapezium.

4. The diagonals $AC$, $BD$ of a parallelogram intersect in $O$, and $P$ is a point within the triangle $AOB$; prove that the difference of the triangles $CPD$, $APD$ is equal to the sum of the triangles $APC$, $BPD$.

5. If either diagonal of a parallelogram be equal to a side of the figure, the other diagonal shall be greater than any side of the figure.

6. If through the angles of a parallelogram four straight lines be drawn parallel to its diagonals, another parallelogram will be formed, the area of which will be double that of the original parallelogram.

7. If two triangles have two sides respectively equal and the included angles supplemental, the triangles are equal.

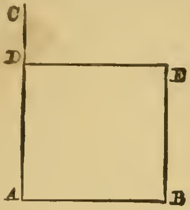
8. Bisect a given triangle by a straight line drawn from a given point in one of the sides.

9. The base $AB$ of a triangle $ABC$ is produced to a point $D$ such that $BD$ is equal to $AB$, and straight lines are drawn from $A$ and $D$ to $E$, the middle point of $BC$; prove that the triangle $ADE$ is equal to the triangle $ABC$.

10. Prove that a pair of the diagonals of the parallelograms, which are about the diameter of any parallelogram, are parallel to each other.

## PROPOSITION XLVI. PROBLEM.

*To describe a square upon a given straight line.*



Let $AB$ be the given st. line.

*It is required to describe a square on $AB$.*

From $A$ draw $AC \perp$ to $AB$. I. 11. Cor.

In $AC$ make $AD=AB$.

Through $D$ draw $DE \parallel$ to $AB$. I. 31.

Through $B$ draw $BE \parallel$ to $AD$. I. 31.

Then $AE$ is a parallelogram,

and $\therefore AB=ED$, and $AD=BE$. I. 34.

But $AB=AD$;

$\therefore AB, BE, ED, DA$ are all equal;

$\therefore AE$ is equilateral.

And $\angle BAD$ is a right angle.

$\therefore AE$ is a square, Def. XXX.

and it is described on $AB$.

Q. E. F.

Ex. 1. Shew how to construct a rectangle whose sides are equal to two given straight lines.

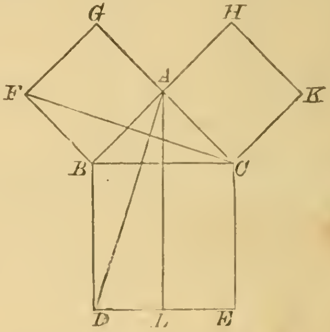
Ex. 2. Shew that the squares on equal straight lines are equal.

Ex. 3. Shew that equal squares must be on equal straight lines.

NOTE. The theorems in Ex. 2 and 3 are assumed by Euclid in the proof of Prop. XLVIII.

### PROPOSITION XLVII. THEOREM.

*In any right-angled triangle the square which is described on the side subtending the right angle is equal to the squares described on the sides which contain the right angle.*



Let $ABC$ be a right-angled $\triangle$, having the rt. $\angle BAC$.

*Then must sq. on $BC =$ sum of sqq. on $BA$, $AC$.*

On $BC$, $CA$, $AB$ descr. the sqq. $BDEC$, $CKHA$, $AGFB$.

Through $A$ draw $AL \parallel$ to $BD$ or $CE$, and join $AD$, $FC$.

Then $\because \angle BAC$ and $\angle BAG$ are both rt. $\angle$s,

$\therefore CAG$ is a st. line; I. 14

and $\because \angle BAC$ and $\angle CAH$ are both rt. $\angle$s;

$\therefore BAH$ is a st. line. I. 14.

Now $\because \angle DBC = \angle FBA$, each being a rt. $\angle$,

adding to each $\angle ABC$, we have

$\angle ABD = \angle FBC$. Ax. 2.

Then in $\triangle$s $ABD$, $FBC$,

$\because AB = FB$, and $BD = BC$, and $\angle ABD = \angle FBC$,

$\therefore \triangle ABD = \triangle FBC$. I. 4.

Now ▱ $BL$ is double of $\triangle ABD$, on same base $BD$ and between same $\parallel$s $AL$, $BD$. I. 41.

and sq. $BG$ is double of $\triangle FBC$, on same base $FB$ and between same $\parallel$s $FB$, $GC$. I. 41.

$\therefore$ ▱ $BL =$ sq. $BG$

Similarly, by joining $AE$, $BK$ it may be shewn that

$$\square\ CL = \text{sq. } AK.$$

Now sq. on $BC$ = sum of $\square$ $BL$ and $\square$ $CL$,

= sum of sq. $BG$ and sq. $AK$,

= sum of sqq. on $BA$ and $AC$.

Q. E. D.

Ex. 1. Prove that the square, described upon the diagonal of any given square, is equal to twice the given square.

Ex. 2. Find a line, the square on which shall be equal to the sum of the squares on three given straight lines.

Ex. 3. If one angle of a triangle be equal to the sum of the other two, and one of the sides containing this angle being divided into four equal parts, the other contains three of those parts; the remaining side of the triangle contains five such parts.

Ex. 4. The triangles $ABC$, $DEF$, having the angles $ACB$, $DFE$ right angles, have also the sides $AB$, $AC$ equal to $DE$, $DF$, each to each; shew that the triangles are equal in every respect.

Note. This Theorem has been already deduced as a Corollary from Prop. E, page 43.

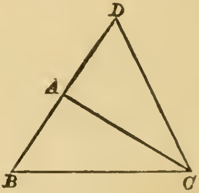
Ex. 5. Divide a given straight line into two parts, so that the square on one part shall be double of the square on the other.

Ex. 6. If from one of the acute angles of a right-angled triangle a line be drawn to the opposite side, the squares on that side and on the line so drawn are together equal to the sum of the squares on the segment adjacent to the right angle and on the hypotenuse.

Ex. 7. In any triangle, if a line be drawn from the vertex at right angles to the base, the difference between the squares on the sides is equal to the difference between the squares on the segments of the base.

### PROPOSITION XLVIII. THEOREM.

*If the square described upon one of the sides of a triangle be equal to the squares described upon the other two sides of it, the angle contained by those sides is a right angle.*



Let the sq. on $BC$, a side of $\triangle ABC$, be equal to the sum of the sqq. on $AB$, $AC$.

*Then must* $\angle BAC$ *be a rt. angle.*

From pt. $A$ draw $AD \perp$ to $AC$. I. 11.

Make $AD = AB$, and join $DC$.

Then $\because AD = AB$,

$\therefore$ sq. on $AD$ = sq. on $AB$; I. 46, Ex. 2.

add to each sq. on $AC$.

then sum of sqq. on $AD$, $AC$ = sum of sqq. on $AB$, $AC$.

But $\because \angle DAC$ is a rt. angle,

$\therefore$ sq. on $DC$ = sum of sqq. on $AD$, $AC$; I. 47.

and, by hypothesis,

sq. on $BC$ = sum of sqq. on $AB$, $AC$;

$\therefore$ sq. on $DC$ = sq. on $BC$;

$\therefore DC = BC$. I. 46, Ex. 3.

Then in $\triangle$s $ABC$, $ADC$,

$\because AB = AD$, and $AC$ is common, and $BC = DC$,

$\therefore \angle BAC = \angle DAC$; I. a

and $\angle DAC$ is a rt. angle, by construction;

$\therefore \angle BAC$ is a rt. angle.

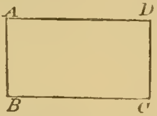
Q. E. D.

# BOOK II.

## INTRODUCTORY REMARKS.

THE geometrical figure with which we are chiefly concerned in this book is the RECTANGLE. A rectangle is said to be *contained by* any two of its adjacent sides.

Thus if *ABCD* be a rectangle, it is said to be contained by *AB*, *AD*, or by any other pair of adjacent sides.



We shall use the abbreviation *rect. AB, AD* to express the words "the rectangle contained by *AB*, *AD*."

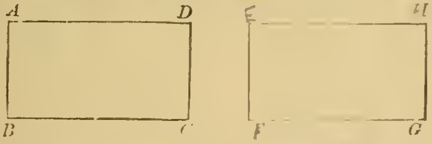
We shall make frequent use of a Theorem (employed, but not demonstrated, by Euclid) which may be thus stated and proved.

### PROPOSITION A. THEOREM.

*If the adjacent sides of one rectangle be equal to the adjacent sides of another rectangle, each to each, the rectangles are equal in area.*

Let *ABCD*, *EFGH* be two rectangles:

and let $AB=EF$ and $BC=FG$.



*Then must rect. ABCD = rect. EFGH.*

For if the rect. *EFGH* be applied to the rect. *ABCD*, so that *EF* coincides with *AB*,

then *FG* will fall on *BC*, $\because \angle EFG = \angle ABC$,

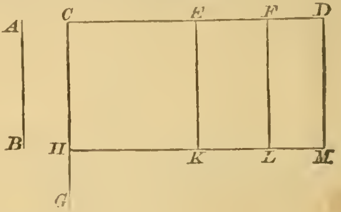
and *G* will coincide with *C*, $\because BC = FG$.

Similarly it may be shewn that *H* will coincide with *D*,

$\therefore$ rect. *EFGH* coincides with and is therefore equal to rect *ABCD*. Q. E. D.

## PROPOSITION I. THEOREM.

*If there be two straight lines, one of which is divided into any number of parts, the rectangle contained by the two straight lines is equal to the rectangles contained by the undivided line and the several parts of the divided line.*



Let $AB$ and $CD$ be two given st. lines,

and let $CD$ be divided into any parts in $E$, $F$.

*Then must rect.* $AB$, $CD$ = *sum of rect.* $AB$, $CE$ *and rect.* $AB$, $EF$ *and rect.* $AB$, $FD$.

From $C$ draw $CG \perp$ to $CD$, and in $CG$ make $CH = AB$.

Through $H$ draw $HM \parallel$ to $CD$. I. 31.

Through $E$, $F$, and $D$ draw $EK$, $FL$, $DM \parallel$ to $CH$.

Then $EK$ and $FL$, being each $= CH$, are each $= AB$.

Now $CM$ = sum of $CK$ and $EL$ and $FM$.

And $CM$ = rect. $AB$, $CD$, $\because CH = AB$,

$CK$ = rect. $AB$, $CE$, $\because CH = AB$,

$EL$ = rect. $AB$, $EF$, $\because EK = AB$,

$FM$ = rect. $AB$, $FD$, $\because FL = AB$;

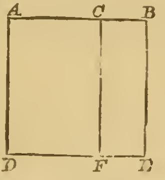
$\therefore$ rect. $AB$, $CD$ = sum of rect. $AB$, $CE$ and rect. $AB$, $EF$ and rect. $AB$, $FD$.

Q. E. D.

Ex. If two straight lines be each divided into any number of parts, the rectangle contained by the two lines is equal to the rectangles contained by all the parts of the one taken separately with all the parts of the other.

## Proposition II. Theorem.

*If a straight line be divided into any two parts, the rectangles contained by the whole and each of the parts are together equal to the square on the whole line.*



Let the st. line $AB$ be divided into any two parts in $C$.

*Then must*

*sq. on $AB$=sum of rect. $AB$, $AC$ and rect. $AB$, $CB$.*

On $AB$ describe the sq. $ADEB$ I. 46.

Through $C$ draw $CF \parallel$ to $AD$. I. 31.

Then $AE$=sum of $AF$ and $CE$.

Now $AE$ is the sq. on $AB$,

$AF$=rect. $AB$, $AC$, $\because AD=AB$,

$CE$=rect. $AB$, $CB$, $\because BE=AB$,

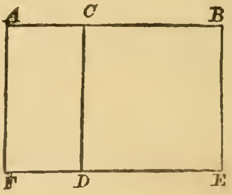
$\therefore$ sq. on $AB$=sum of rect. $AB$, $AC$ and rect. $AB$, $CB$.

Q. E. D.

Ex. The square on a straight line is equal to four times the square on half the line.

## PROPOSITION III. THEOREM.

*If a straight line be divided into any two parts, the rectangle contained by the whole and one of the parts is equal to the rectangle contained by the two parts together with the square on the aforesaid part.*



Let the st. line $AB$ be divided into any two parts in $C$.

*Then must*

*rect.* $AB$, $CB$=*sum of rect.* $AC$, $CB$ *and sq. on* $CB$.

On $CB$ describe the sq. $CDEB$. I. 46

From $A$ draw $AF \parallel$ to $CD$, meeting $ED$ produced in $F$.

Then $AE$=sum of $AD$ and $CE$.

Now $AE$=rect. $AB$, $CB$, $\because BE=CB$,

$AD$=rect. $AC$, $CB$, $\because CD=CB$,

$CE$=sq. on $CB$.

$\therefore$ rect. $AB$, $CB$=sum of rect. $AC$, $CB$ and sq. on $CB$.

Q. E. D.

NOTE. When a straight line is cut in a point, the distances of the point of section from the ends of the line are called the *segments* of the line.

If a line $AB$ be divided in $C$,

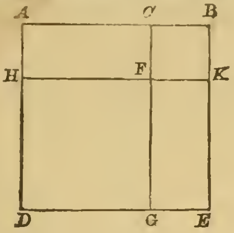
$AC$ and $CB$ are called the *internal* segments of $AB$.

If a line $AC$ be produced to $B$,

$AB$ and $CB$ are called the *external* segments of $AC$.

## PROPOSITION IV. THEOREM.

*If a straight line be divided into any two parts, the square on the whole line is equal to the squares on the two parts together with twice the rectangle contained by the parts.*



Let the st. line $AB$ be divided into any two parts in $C$.

*Then must*

*sq. on $AB$=sum of sqq. on $AC$, $CB$ and twice rect. $AC$, $CB$.*

On $AB$ describe the sq. $ADEB$. I. 46.

From $AD$ cut off $AH=CB$. Then $HD=AC$. a + 3

Draw $CG \parallel$ to $AD$, and $HK \parallel$ to $AB$, meeting $CG$ in $F$.

Then $\because BK=AH$, $\therefore BK=CB$, Ax. 1.

$\therefore BK$, $KF$, $FC$, $CB$ are all equal ; and $KBC$ is a rt. $\angle$ ;

$\therefore CK$ is the sq. on $CB$. Def. xxx.

Also $HG$=sq. on $AC$, $\because HF$ and $HD$ each$=AC$.

Now $AE$=sum of $HG$, $CK$, $AF$, $FE$,

and $AE$=sq. on $AB$,
$HG$=sq. on $AC$,
$CK$=sq. on $CB$,
$AF$=rect. $AC$, $CB$, $\because CF=CB$,
$FE$=rect. $AC$, $CB$, $\because FG=AC$ and $FK=CB$.

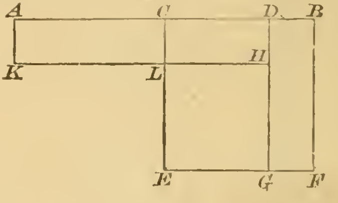
$\therefore$ sq. on $AB$=sum of sqq. on $AC$, $CB$ and twice rect. $AC$, $CB$.

Q. E. D.

Ex. In a triangle, whose vertical angle is a right angle, a straight line is drawn from the vertex perpendicular to the base. Shew that the rectangle, contained by the segments of the base, is equal to the square on the perpendicular.

## PROPOSITION V. THEOREM.

*If a straight line be divided into two equal parts and also into two unequal parts, the rectangle contained by the unequal parts, together with the square on the line between the points of section, is equal to the square on half the line.*



Let the st. line $AB$ be divided equally in $C$ and unequally in $D$.

*Then must*

*rect. $AD$, $DB$ together with sq. on $CD$ = sq. on $CB$.*

| | |
|---|---|
| On $CB$ describe the sq. $CEFB$. | I. 46. |
| Draw $DG \parallel$ to $CE$, and from it cut off $DH = DB$. | I. 31. |
| Draw $HLK \parallel$ to $AD$, and $AK \parallel$ to $DH$. | I. 31. |

Then rect. $DF$ = rect. $AL$, $\because BF = AC$, and $BD = CL$.

Also $LG$ = sq. on $CD$, $\because LH = CD$, and $HG = CD$.

Then rect. $AD$, $DB$ together with sq. on $CD$

= $AH$ together with $LG$

= sum of $AL$ and $CH$ and $LG$

= sum of $DF$ and $CH$ and $LG$

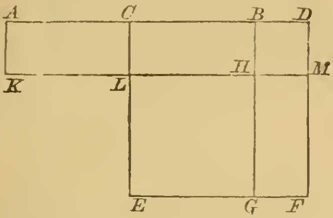
= $CF$

= sq. on $CB$.

Q. E. D.

## PROPOSITION VI. THEOREM.

*If a straight line be bisected and produced to any point, the rectangle contained by the whole line thus produced and the part of it produced, together with the square on half the line bisected, is equal to the square on the straight line which is made up of the half and the part produced.*



Let the st. line $AB$ be bisected in $C$ and produced to $D$.

*Then must*

*rect. $AD$, $DB$ together with sq. on $CB$ = sq. on $CD$.*

On $CD$ describe the sq. $CEFD$. I. 46.

Draw $BG$ ∥ to $CE$, and cut off $BH = BD$. I. 31

Through $H$ draw $KLM$ ∥ to $AD$ I. 31.

Through $A$ draw $AK$ ∥ to $CE$.

Now ∵ $BG = CD$ and $BH = BD$;

∴ $HG = CB$; Ax. 3.

∴ rect. $MG$ = rect. $AL$. II. A.

Then rect. $AD$, $DB$ together with sq. on $CB$

= sum of $AM$ and $LG$

= sum of $AL$ and $CM$ and $LG$

= sum of $MG$ and $CM$ and $LG$

= $CF$,

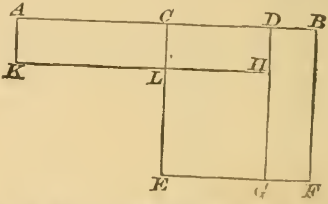
= sq. on $CD$.

Q. E. D.

NOTE. We here give the proof of an important theorem, which is usually placed as a corollary to Proposition V.

### PROPOSITION B. THEOREM.

*The difference between the squares on any two straight lines is equal to the rectangle contained by the sum and difference of those lines.*



Let $AC$, $CD$ be two st. lines, of which $AC$ is the greater, and let them be placed so as to form one st. line $AD$.

Produce $AD$ to $B$, making $CB=AC$.

Then $AD=$ the sum of the lines $AC$, $CD$,

and $DB=$ the difference of the lines $AC$, $CD$.

*Then must difference between sqq. on* $AC$, $CD=$ *rect.* $AD$, $DB$.

On $CB$ describe the sq. $CEFB$ I. 46.

Draw $DG \parallel$ to $CE$, and from it cut off $DH=DB$. I. 31.

Draw $HLK \parallel$ to $AD$, and $AK \parallel$ to $DH$ I. 31.

Then rect. $DF=$ rect. $AL$, $\because BF=AC$, and $BD=CL$.

Also $LG=$ sq. on $CD$, $\because LH=CD$, and $HG=CD$.

Then difference between sqq. on $AC$, $CD$

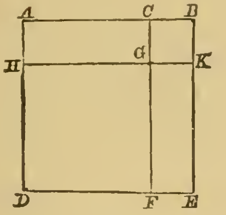
$=$ difference between sqq. on $CB$, $CD$
$=$ sum of $CH$ and $DF$
$=$ sum of $CH$ and $AL$
$=AH$
$=$ rect. $AD$, $DH$
$=$ rect. $AD$, $DB$.

Q. E. D.

Ex. Shew that Propositions V. and VI. might be deduced from this Proposition.

## PROPOSITION VII. THEOREM.

*If a straight line be divided into any two parts, the squares on the whole line and on one of the parts are equal to twice the rectangle contained by the whole and that part together with the square on the other part.*



Let $AB$ be divided into any two parts in $C$.

*Then must*

*sqq. on $AB$, $BC$ = twice rect. $AB$, $BC$ together with sq. on $AC$.*

On $AB$ describe the sq. $ADEB$. I. 46.

From $AD$ cut off $AH = CB$.

Draw $CF \parallel$ to $AD$ and $HGK \parallel$ to $AB$. I. 31.

Then $HF$ = sq. on $AC$, and $CK$ = sq. on $CB$.

Then sqq. on $AB$, $BC$ = sum of $AE$ and $CK$

= sum of $AK$, $HF$, $GE$ and $CK$

= sum of $AK$, $HF$ and $CE$.

Now $AK$ = rect. $AB$, $BC$, $\because BK = BC$;

$CE$ = rect. $AB$, $BC$, $\because BE = AB$;

$HF$ = sq. on $AC$.

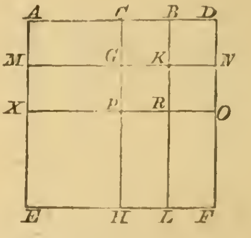
$\therefore$ sqq. on $AB$, $BC$ = twice rect. $AB$, $BC$ together with sq. on $AC$.

Q. E. D.

Ex. If straight lines be drawn from $G$ to $B$ and from $G$ to $D$, shew that $BGD$ is a straight line.

## PROPOSITION VIII. THEOREM.

*If a straight line be divided into any two parts, four times the rectangle contained by the whole line and one of the parts, together with the square on the other part, is equal to the square on the straight line which is made up of the whole and the first part.*



Let the st. line $AB$ be divided into any two parts in $C$.

Produce $AB$ to $D$, so that $BD = BC$.

*Then must four times rect.* $AB$, $BC$ *together with sq. on* $AC$ = *sq. on* $AD$.

On $AD$ describe the sq. $AEFD$. I. 46.

From $AE$ cut off $AM$ and $MX$ each $= CB$.

Through $C$, $B$ draw $CH$, $BL$ ∥ to $AE$. I. 31.

Through $M$, $X$ draw $MGKN$, $XPRO$ ∥ to $AD$. I. 31.

Now ∵ $XE = AC$, and $XP = AC$, ∴ $XH$ = sq. on $AC$.

Also $AG = MP = PL = RF$, II. A.

and $CK = GR = BN = KO$; II. A.

∴ sum of these eight rectangles

= four times the sum of $AG$, $CK$

= four times $AK$

= four times rect. $AB$, $BC$.

Then four times rect. $AB$, $BC$ and sq. on $AC$

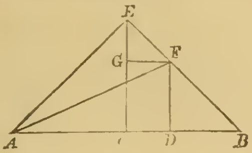
= sum of the eight rectangles and $XH$

= $AEFD$

= sq. on $AD$. Q. E. D.

### PROPOSITION IX. THEOREM.

*If a straight line be divided into two equal, and also into two unequal parts, the squares on the two unequal parts are together double of the square on half the line and of the square on the line between the points of section.*



Let $AB$ be divided equally in $C$ and unequally in $D$.

*Then must*

*sum of sqq. on $AD$, $DB$ = twice sum of sqq. on $AC$, $CD$.*

Draw $CE = AC$ at rt. $\angle$s to $AB$, and join $EA$, $EB$.

Draw $DF$ at rt. $\angle$s to $AB$, meeting $EB$ in $F$.

Draw $FG$ at rt. $\angle$s to $EC$, and join $AF$.

Then $\because$ $\angle ACE$ is a rt. $\angle$,

$\therefore$ sum of $\angle$s $AEC$, $EAC$ = a rt. $\angle$; I. 32.

and $\because$ $\angle AEC = \angle EAC$, I. A.

$\therefore$ $\angle AEC$ = half a rt. $\angle$.

So also $\angle BEC$ and $\angle EBC$ are each = half a rt. $\angle$.

Hence $\angle AEF$ is a rt. $\angle$.

Also, $\because$ $\angle GEF$ is half a rt. $\angle$, and $\angle EGF$ is a rt. $\angle$;

$\therefore$ $\angle EFG$ is half a rt. $\angle$;

$\therefore$ $\angle EFG = \angle GEF$, and $\therefore$ $EG = GF$. I. B. Cor.

So also $\angle BFD$ is half a rt. $\angle$, and $BD = DF$.

Now sum of sqq. on $AD$, $DB$

= sq. on $AD$ together with sq. on $DF$

= sq. on $AF$ I. 47.

= sq. on $AE$ together with sq. on $EF$ I. 47.

= sqq. on $AC$, $EC$ together with sqq. on $EG$, $GF$ I. 47.

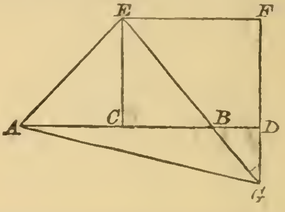
= twice sq. on $AC$ together with twice sq. on $GF$

= twice sq. on $AC$ together with twice sq. on $CD$.

Q. E. D.

## PROPOSITION X. THEOREM.

*If a straight line be bisected and produced to any point, the square on the whole line thus produced and the square on the part of it produced are together double of the square on half the line bisected and of the square on the line made up of the half and the part produced.*



Let the st. line $AB$ be bisected in $C$ and produced to $D$.

*Then must*

*sum of sqq. on* $AD$, $BD$ = *twice sum of sqq. on* $AC$, $CD$.

Draw $CE \perp$ to $AB$, and make $CE = AC$.

Join $EA$, $EB$ and draw $EF \parallel$ to $AD$ and $DF \parallel$ to $CE$.

Then $\because \angle$s $FEB$, $EFD$ are together less than two rt. $\angle$s,

$\therefore$ $EB$ and $FD$ will meet if produced towards $B$, $D$ in some pt. $G$.

Join $AG$.

Then $\because \angle ACE$ is a rt. $\angle$,

$\therefore$ $\angle$s $EAC$, $AEC$ together = a rt. $\angle$,

and $\because \angle EAC = \angle AEC$, I. A.

$\therefore \angle AEC$ = half a rt. $\angle$.

So also $\angle$s $BEC$, $EBC$ each = half a rt. $\angle$.

$\therefore \angle AEB$ is a rt. $\angle$.

Also $\angle DBG$, which = $\angle EBC$, is half a rt. $\angle$,

and $\therefore \angle BGD$ is half a rt. $\angle$;

$\therefore BD = DG$. I. B. Cor.

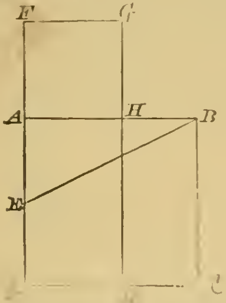
Again, $\because \angle FGE$ = half a rt. $\angle$, and $\angle EFG$ is a rt. $\angle$, I. 34.

$\therefore \angle FEG$ = half a rt. $\angle$, and $EF = FG$. I. B. Cor.

Then sum of sqq. on $AD$, $DB$
= sum of sqq. on $AD$, $DG$
= sq. on $AG$ I. 47.
= sq. on $AE$ together with sq. on $EG$ I. 47.
= sqq. on $AC$, $EC$ together with sqq. on $EF$, $FG$ I. 47.
= twice sq. on $AC$ together with twice sq. on $EF$
= twice sq. on $AC$ together with twice sq. on $CD$. Q. E. D.

## PROPOSITION XI. PROBLEM.

*To divide a given straight line into two parts, so that the rectangle contained by the whole and one of the parts shall be equal to the square on the other part.*



Let $AB$ be the given st. line.

On $AB$ descr. the sq. $ADCB$. I. 46.

Bisect $AD$ in $E$ and join $EB$. I. 10.

Produce $DA$ to $F$, making $EF=EB$.

On $AF$ descr. the sq. $AFGH$. I. 46.

*Then AB is divided in H so that rect. AB, BH = sq. on AH.*

Produce $GH$ to $K$.

Then $\because$ $DA$ is bisected in $E$ and produced to $F$,
$\therefore$ rect. $DF$, $FA$ together with sq. on $AE$
= sq. on $EF$ II. 6.
= sq. on $EB$, $\because$ $EB=EF$,
= sum of sqq. on $AB$, $AE$. I. 47.

Take from each the square on $AE$.

Then rect. $DF$, $FA$ = sq. on $AB$. Ax. 3.

Now $FK$ = rect. $DF$, $FA$, $\because$ $FG=FA$.

$\therefore$ $FK=AC$.

Take from each the common part $AK$.

Then $FH=HC$;

that is, sq. on $AH$ = rect. $AB$, $BH$, $\because$ $BC=AB$.

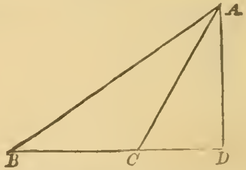
Thus $AB$ is divided in $H$ as was reqd.

Q. E. F.

Ex. Shew that the squares on the whole line and one of the parts are equal to three times the square on the other part.

### PROPOSITION XII. THEOREM.

*In obtuse-angled triangles, if a perpendicular be drawn from either of the acute angles to the opposite side produced, the square on the side subtending the obtuse angle is greater than the squares on the sides containing the obtuse angle, by twice the rectangle contained by the side, upon which, when produced, the perpendicular falls, and the straight line intercepted without the triangle between the perpendicular and the obtuse angle.*



Let $ABC$ be an obtuse-angled △, having ∠ $ACB$ obtuse.

From $A$ draw $AD$ ⊥ to $BC$ produced.

*Then must sq. on AB be greater than sum of sqq. on BC, CA by twice rect. BC, CD.*

For since $BD$ is divided into two parts in $C$,
sq. on $BD$=sum of sqq. on $BC$, $CD$, and twice rect. $BC$, $CD$. II. 4.

Add to each sq. on $DA$: then
sum of sqq. on $BD$, $DA$=sum of sqq. on $BC$, $CD$, $DA$ and twice rect. $BC$, $CD$.

Now sqq. on $BD$, $DA$=sq. on $AB$, I. 47.
and sqq. on $CD$, $DA$=sq. on $CA$; I. 47.

∴ sq. on $AB$=sum of sqq. on $BC$, $CA$ and twice rect. $BC$, $CD$.

∴ sq. on $AB$ is greater than sum of sqq. on $BC$, $CA$ by twice rect. $BC$, $CD$.

Q. E. D.

Ex. The squares on the diagonals of a trapezium are together equal to the squares on its two sides, which are not parallel, and twice the rectangle contained by the sides, which are parallel.

## PROPOSITION XIII. THEOREM.

*In every triangle, the square on the side subtending any of the acute angles is less than the squares on the sides containing that angle, by twice the rectangle contained by either of these sides and the straight line intercepted between the perpendicular, let fall upon it from the opposite angle, and the acute angle.*

FIG. 1.

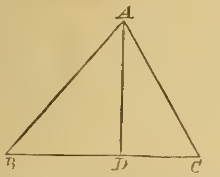
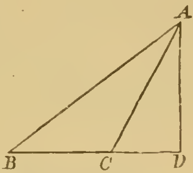


FIG. 2.



Let $ABC$ be any $\triangle$, having the $\angle ABC$ acute.

From $A$ draw $AD \perp$ to $BC$ or $BC$ produced.

*Then must sq. on $AC$ be less than the sum of sqq. on $AB$, $BC$, by twice rect. $BC$, $BD$.*

For in Fig. 1 $BC$ is divided into two parts in $D$, and in Fig. 2 $BD$ is divided into two parts in $C$;

$\therefore$ in both cases

sum of sqq. on $BC$, $BD$ = sum of twice rect. $BC$, $BD$ and sq. on $CD$. II. 7.

Add to each the sq. on $DA$, then

sum of sqq. on $BC$, $BD$, $DA$ = sum of twice rect. $BC$, $BD$ and sqq. on $CD$, $DA$;

$\therefore$ sum of sqq. on $BC$, $AB$ = sum of twice rect. $BC$, $BD$ and sq. on $AC$; I. 47.

$\therefore$ sq. on $AC$ is less than sum of sqq. on $AB$, $BC$ by twice rect. $BC$, $BD$.

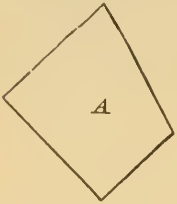
The case, in which the perpendicular $AD$ coincides with $AC$, needs no proof.

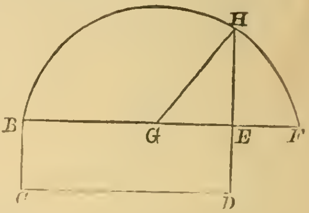
Q. E. D.

Ex. Prove that the sum of the squares on any two sides of a triangle is equal to twice the sum of the squares on half the base and on the line joining the vertical angle with the middle point of the base.

### PROPOSITION XIV. PROBLEM.

*To describe a square that shall be equal to a given rectilinear figure.*





Let $A$ be the given rectil. figure.
*It is reqd. to describe a square that shall* $=A$.

Describe the rectangular ▱ $BCDE=A$. I. 45.
Then if $BE=ED$ the ▱ $BCDE$ is a square,
and what was reqd. is done.

But if $BE$ be not $=ED$, produce $BE$ to $F$, so that $EF=ED$.
Bisect $BF$ in $G$; and with centre $G$ and distance $GB$,
describe the semicircle $BHF$.
Produce $DE$ to $H$ and join $GH$.

Then, ∵ $BF$ is divided equally in $G$ and unequally in $E$,
∴ rect. $BE$, $EF$ together with sq. on $GE$
$=$sq. on $GF$ II. 5.
$=$sq. on $GH$
$=$sum of sqq. on $EH$, $GE$. I. 47.
Take from each the square on $GE$.
Then rect. $BE$, $EF=$sq. on $EH$.
But rect. $BE$, $EF=BD$, ∵ $EF=ED$;
∴ sq. on $EH=BD$;
∴ sq. on $EH=$rectil. figure $A$.

Q. E. F.

## *Miscellaneous Exercises on Book II.*

1. In a triangle, whose vertical angle is a right angle, a straight line is drawn from the vertex perpendicular to the base; shew that the square on either of the sides adjacent to the right angle is equal to the rectangle contained by the base and the segment of it adjacent to that side.

2. The squares on the diagonals of a parallelogram are together equal to the squares on the four sides.

3. If $ABCD$ be any rectangle, and $O$ any point either within or without the rectangle, shew that the sum of the squares on $OA$, $OC$ is equal to the sum of the squares on $OB$, $OD$.

4. If either diagonal of a parallelogram be equal to one of the sides about the opposite angle of the figure, the square on it shall be less than the square on the other diameter, by twice the square on the other side about that opposite angle.

5. Produce a given straight line $AB$ to $C$, so that the rectangle, contained by the sum and difference of $AB$ and $AC$, may be equal to a given square.

6. Shew that the sum of the squares on the diagonals of any quadrilateral is less than the sum of the squares on the four sides, by four times the square on the line joining the middle points of the diagonals.

7. If the square on the perpendicular from the vertex of a triangle is equal to the rectangle, contained by the segments of the base, the vertical angle is a right angle.

8. If two straight lines be given, shew how to produce one of them so that the rectangle contained by it and the produced part may be equal to the square on the other.

9. If a straight line be divided into three parts, the square on the whole line is equal to the sum of the squares on the parts together with twice the rectangle contained by each two of the parts.

10. In any quadrilateral the squares on the diagonals are together equal to twice the sum of the squares on the straight lines joining the middle points of opposite sides.

11. If straight lines be drawn from each angle of a triangle to bisect the opposite sides, four times the sum of the squares on these lines is equal to three times the sum of the squares on the sides of the triangle.

12. *CD* is drawn perpendicular to *AB*, a side of the triangle *ABC*, in which *AC*=*AB*. Shew that the square on *CD* is equal to the square on *BD* together with twice the rectangle *AD*, *DB*.

13. The hypotenuse *AB* of a right-angled triangle *ABC* is trisected in the points *D*, *E*; prove that if *CD*, *CE* be joined, the sum of the squares on the sides of the triangle *CDE* is equal to two-thirds of the square on *AB*.

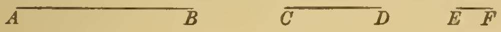
14. The square on the hypotenuse of an isosceles right angled triangle is equal to four times the square on the perpendicular from the right angle on the hypotenuse.

15. Divide a given straight line into two parts, so that the rectangle contained by them shall be equal to the square described upon a straight line, which is less than half the line divided.

## NOTE 6.—*On the Measurement of Areas.*

*To measure* a Magnitude we fix upon some magnitude of the same kind to serve as a standard or unit; and then any magnitude of that kind is measured by the number of times it contains this unit, and this number is called the MEASURE of the quantity.

Suppose, for instance, we wish to measure a straight line $AB$. We take another straight line $EF$ for our standard,



and then we say

if $AB$ contain $EF$ three times, the measure of $AB$ is 3,  
if ....................four........................................4,  
if .................... $x$ ........................................$x$.

Next suppose we wish to measure two straight lines $AB$, $CD$ by the same standard $EF$.

If $AB$ contain $EF$ $m$ times  
and $CD$ .............. $n$ times,

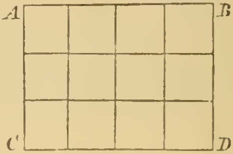
where $m$ and $n$ stand for numbers, whole or fractional, we say that $AB$ and $CD$ are *commensurable.*

But it may happen that we may be able to find a standard line $EF$, such that it is contained an exact number of times in $AB$; and yet there is no number, whole or fractional, which will express the number of times $EF$ is contained in $CD$.

In such a case, where no unit-line can be found, such that it is contained an exact number of times in *each* of two lines $AB$, $CD$, these two lines are called *incommensurable.*

In the processes of Geometry we constantly meet with incommensurable magnitudes. Thus the side and diagonal of a square are incommensurables; and so are the diameter and circumference of a circle.

Next, suppose two lines $AB$, $AC$ to be at right angles to each other and to be commensurable, so that $AB$ contains four times a certain unit of linear measurement, which is contained by $AC$ three times.



Divide $AB$, $AC$ into four and three equal parts respectively, and draw lines through the points of division parallel to $AC$, $AB$ respectively; then the rectangle $ACDB$ is divided into a number of equal squares, each constructed on a line equal to the unit of linear measurement.

If one of these squares be taken as the unit of area, the *measure* of the area of the rectangle $ACDB$ will be the number of these squares.

Now this number will evidently be the same as that obtained by multiplying the measure of $AB$ by the measure of $AC$; that is, the measure of $AB$ being 4 and the measure of $AC$ 3, the measure of $ACDB$ is $4 \times 3$ or 12. (Algebra, Art. 38.)

And *generally*, if the measures of two adjacent sides of a rectangle, supposed to be commensurable, be $a$ and $b$, then the measure of the rectangle will be $ab$. (Algebra, Art. 39.)

If all lines were commensurable, then, whatever might be the length of two adjacent sides of a rectangle, we might select the unit of length, so that the measures of the two sides should be whole numbers; and then we might apply the processes of Algebra to establish many Propositions in Geometry by simpler methods than those adopted by Euclid.

Take, for example, the theorem in Book II. Prop. IV.

If all lines were commensurable we might proceed thus —

Let the measure of $AC$ be $x$,
..................... of $CB$ ... $y$,
Then the measure of $AB$ is $x+y$.

Now $$(x+y)^2 = x^2 + y^2 + 2xy,$$

which proves the theorem.

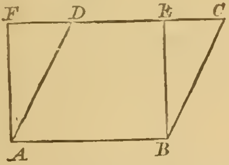
But, inasmuch as all lines are not commensurable, we have in Geometry to treat of *magnitudes* and not of *measures:* that is, when we use the symbol $A$ to represent a line (as in I. 22), $A$ stands for the line itself and not, as in Algebra, for the number of units of length contained by the line.

The method, adopted by Euclid in Book II. to explain the relations between the rectangles contained by certain lines, is more exact than any method founded upon Algebraical principles can be ; because his method applies not merely to the case in which the sides of a rectangle are commensurable, but also to the case in which they are incommensurable.

The student is now in a position to understand the practical application of the theory of Equivalence of Areas, of which the foundation is the 35th Proposition of Book I. We shall give a few examples of the use made of this theory in Mensuration.

### *Area of a Parallelogram.*

The area of a parallelogram $ABCD$ is equal to the area of the rectangle $ABEF$ on the same base $AB$ and between the same parallels $AB$, $FC$.



Now $BE$ is the altitude of the parallelogram $ABCD$ if $AB$ be taken as the base.

Hence area of ▱ $ABCD$ = rect. $AB$, $BE$.
If then the measure of the base be denoted by $b$,
and .......................... altitude ............ $h$,
the measure of the area of the ▱ will be denoted by $bh$
That is, when the base and altitude are commensurable,
measure of area = measure of base into measure of altitude.

### *Area of a Triangle.*

If from one of the angular points $A$ of a triangle $ABC$, a perpendicular $AD$ be drawn to $BC$, Fig. 1, or to $BC$ produced, Fig. 2,

FIG. 1.

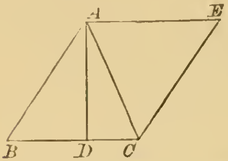
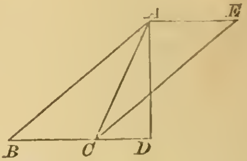


FIG. 2.



and if, in both cases, a parallelogram $ABCE$ be completed of which $AB$, $BC$ are adjacent sides,

area of $\triangle\ ABC =$ half of area of $\square\ ABCE$.

Now if the measure of $BC$ be $b$,
and ........................$AD$... $h$,

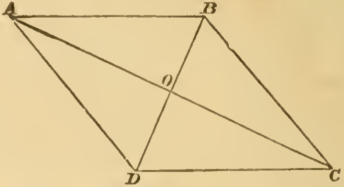
measure of area of $\square\ ABCE$ is $bh$;

$\therefore$ measure of area of $\triangle\ ABC$ is $\frac{bh}{2}$.

### *Area of a Rhombus.*

Let $ABCD$ be the given rhombus.

Draw the diagonals $AC$ and $BD$, cutting one another in $O$.



It is easy to prove that $AC$ and $BD$ bisect each other at right angles.

Then if the measure of $AC$ be $x$,
and ...................... $BD$ ... $y$,

measure of area of rhombus = twice measure of $\triangle\ ACD$.

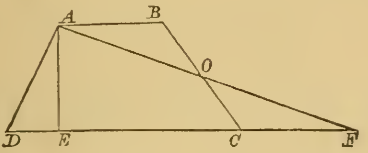
$$= \text{twice } \frac{xy}{4}$$

$$= \frac{xy}{2}.$$

## *Area of a Trapezium.*

Let $ABCD$ be the given trapezium, having the sides $AB$, $CD$ parallel.

Draw $AE$ at right angles to $CD$.



Produce $DC$ to $F$, making $CF=AB$.

Join $AF$, cutting $BC$ in $O$.

Then in $\triangle$s $AOB$, $COF$,

$\because \angle BAO = \angle CFO$, and $\angle AOB = \angle FOC$, and $AB = CF$;

$\therefore \triangle COF = \triangle AOB$. I. 26.

Hence trapezium $ABCD = \triangle ADF$.

Now suppose the measures of $AB$, $CD$, $AE$ to be $m$, $n$, $p$ respectively;

$\therefore$ measure of $DF = m + n$, $\because CF = AB$.

Then measure of area of trapezium

$= \frac{1}{2}$ (measure of $DF \times$ measure of $AE$)

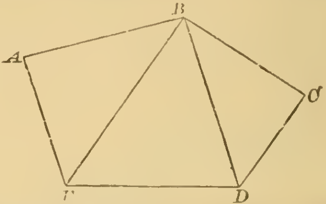
$= \frac{1}{2}(m+n) \times p$.

That is, the measure of the area of a trapezium is found by multiplying half the measure of the sum of the parallel sides by the measure of the perpendicular distance between the parallel sides.

### *Area of an Irregular Polygon.*

There are three methods of finding the area of an irregular polygon, which we shall here briefly notice.

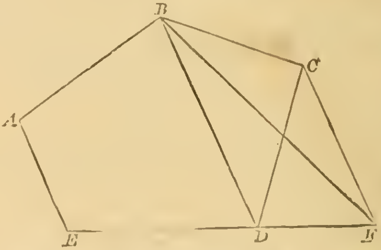
I. *The polygon may be divided into triangles*, and the area of each of these triangles be found separately.



Thus the area of the irregular polygon *ABCDE* is equal to the sum of the areas of the triangles *ABE*, *EBD*, *DBC*.

II. *The polygon may be converted into a single triangle of equal area.*

If *ABCDE* be a pentagon, we can convert it into an equivalent quadrilateral by the following process:



Join *BD* and draw *CF* parallel to *BD*, meeting *ED* produced in *F*, and join *BF*.

Then will quadrilateral *ABFE*=pentagon *ABCDE*.

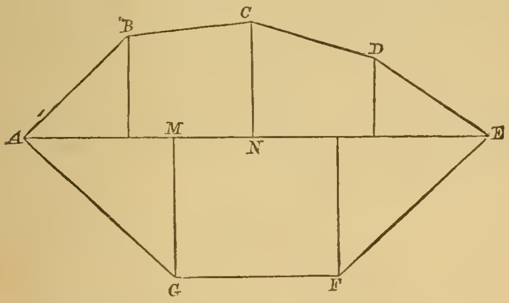
For △ *BDF*= △ *BCD*, on same base *BD* and between same parallels.

If, then, from the pentagon we remove △ *BCD*, and add △ *BDF* to the remainder, we obtain a quadrilateral *ABFE* equivalent to the pentagon *ABCDE*.

The quadrilateral may then, by a similar process, be converted into an equivalent triangle, and thus a polygon of any number of sides may be gradually converted into an equivalent triangle.

The area of this triangle may then be found.

III. The third method is chiefly employed in practice by Surveyors



Let *ABCDEFG* be an irregular polygon.

Draw *AE*, the longest diagonal, and drop perpendiculars on *AE* from the other angular points of the polygon.

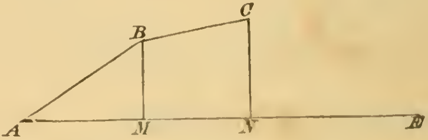
The polygon is thus divided into figures which are either right-angled triangles, rectangles, or trapeziums ; and the areas of each of these figures may be readily calculated.

## Note 7. *On Projections.*

The projection of a *point* $B$, on a straight line of unlimited length $AE$, is the point $M$ at the foot of the perpendicular dropped from $B$ on $AE$.

The projection of a *straight line* $BC$, on a straight line of unlimited length $AE$, is $MN$,—the part of $AE$ intercepted between perpendiculars drawn from $B$ and $C$.

When two lines, as $AB$ and $AE$, form an angle, the projection of $AB$ on $AE$ is $AM$.



We might employ the term projection with advantage to shorten and make clearer the enunciations of Props. XII. and XIII. of Book II.

Thus the enunciation of Prop. XII. might be :—

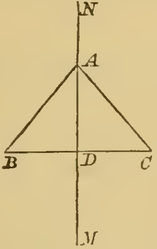
"In oblique-angled triangles, the square on the side subtending the obtuse angle is greater than the squares on the sides containing that angle, by twice the rectangle contained by one of these sides and the projection of the other on it."

The enunciation of Prop. XIII. might be altered in a similar manner.

## NOTE 8. *On Loci.*

Suppose we have to determine the position of a point, which is equidistant from the extremities of a given straight line *BC*.

There is an infinite number of points satisfying this condition, for the vertex of any isosceles triangle, described on *BC* as its base, is equidistant from *B* and *C*.



Let *ABC* be *one* of the isosceles triangles described on *BC*.

If *BC* be bisected in *D*, *MN*, a perpendicular to *BC* drawn through *D*, will pass through *A*.

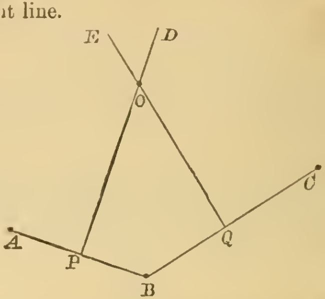
It is easy to shew that any point in *MN*, or *MN* produced in either direction, is equidistant from *B* and *C*.

It may also be proved that no point out of *MN* is equidistant from *B* and *C*.

The line *MN* is called the Locus of all the points, infinite in number, which are equidistant from *B* and *C*.

Def. In plane Geometry *Locus* is the name given to a line, straight or curved, all of whose points satisfy a certain geometrical condition (or have a common property), to the exclusion of all other points.

Next, suppose we have to determine the position of a point, which is equidistant from three given points *A*, *B*, *C*, not in the same straight line.



If we join *A* and *B*, we know that all points equidistant from *A* and *B* lie in the line *PD*, which bisects *AB* at right angles.

If we join *B* and *C*, we know that all points equidistant from *B* and *C* lie in the line *QE*, which bisects *BC* at right angles.

Hence *O*, the point of intersection of *PD* and *QE*, is the only point equidistant from *A*, *B* and *C*.

*PD* is the Locus of points equidistant from *A* and *B*,
*QE*.......................................................*B* and *C*,
and the Intersection of these Loci determines the point. which is equidistant from *A*, *B* and *C*.

*Examples of Loci.*

Find the loci of

(1) Points at a given distance from a given point.

(2) Points at a given distance from a given straight line.

(3) The middle points of straight lines drawn from a given point to a given straight line.

(4) Points equidistant from the arms of an angle.

(5) Points equidistant from a given circle.

(6) Points equally distant from two straight lines which intersect.

## NOTE 9. *On the Methods employed in the solution of Problems.*

In the solution of Geometrical Exercises, certain methods may be applied with success to particular classes of questions.

We propose to make a few remarks on these methods, so far as they are applicable to the first two books of Euclid's Elements.

### *The Method of Synthesis.*

In the Exercises, attached to the Propositions in the preceding pages, the construction of the diagram, necessary for the solution of each question, has usually been fully described, or sufficiently suggested.

The student has in most cases been required simply to apply the geometrical fact, proved in the Proposition preceding the exercise, in order to arrive at the conclusion demanded in the question.

This way of proceeding is called Synthesis (σύνθεσις = composition), because in it we proceed by a regular chain of reasoning from what is *given* to what is *sought.* This being the method employed by Euclid throughout the Elements, we have no need to exemplify it here.

### *The Method of Analysis.*

The solution of many Problems is rendered more easy *by supposing the problem solved and the diagram constructed.* It is then often possible to observe relations between lines, angles and figures in the diagram, which are suggestive of the steps by which the necessary construction might have been effected.

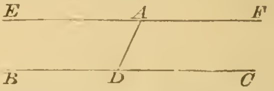
This is called the Method of Analysis (ἀνάλυσις = resolution). It is a method of discovering truth by reasoning concerning things unknown or propositions merely supposed, as if the one were given or the other were really true. The process can best be explained by the following examples.

Our first example of the Analytical process shall be the 31st Proposition of Euclid's First Book.

Ex. 1. *To draw a straight line through a given point parallel to a given straight line.*

Let $A$ be the given point, and $BC$ be the given straight line.

Suppose the problem to be effected, and $EF$ to be the straight line required.



Now we know that any straight line $AD$ drawn from $A$ to meet $BC$ makes equal angles with $EF$ and $BC$. (I. 29.)

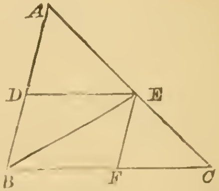
This is a fact from which we can work backward, and arrive at the steps necessary for the solution of the problem ; thus :

Take any point $D$ in $BC$, join $AD$, make $\angle EAD = \angle ADC$, and produce $EA$ to $F$: then $EF$ must be parallel to $BC$.

Ex. 2. *To inscribe in a triangle a rhombus, having one of its angles coincident with an angle of the triangle.*

Let $ABC$ be the given triangle.

Suppose the problem to be effected, and $DBFE$ to be the rhombus.



Then if $EB$ be joined, $\angle DBE = \angle FBE$.

This is a fact from which we can work backward, and deduce the necessary construction ; thus :

Bisect $\angle ABC$ by the straight line $BE$, meeting $AC$ in $E$.

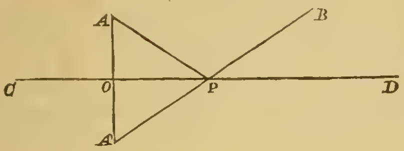
Draw $ED$ and $EF$ parallel to $BC$ and $AB$ respectively.

Then $DBFE$ is the rhombus required. (See Ex. 4, p. 59.)

Ex. 3. *To determine the point in a given straight line, at which straight lines, drawn from two given points, on the same side of the given line, make equal angles with it.*

Let $CD$ be the given line, and $A$ and $B$ the given points.

Suppose the problem to be effected, and $P$ to be the point required.



We then reason thus :

If $BP$ were produced to some point $A$ ,

$\angle CPA'$, being $= \angle BPD$, will be $= \angle APC$.

Again, if $PA'$ be made equal to $PA$,

$AA'$ will be bisected by $CP$ at right angles.

This is a fact from which we can work backward, and find the steps necessary for the solution of the problem ; thus :

From $A$ draw $AO \perp$ to $CD$.

Produce $AO$ to $A'$, making $OA' = OA$.

Join $BA'$, cutting $CD$ in $P$.

Then $P$ is the point required.

## NOTE 10. *On Symmetry.*

The problem, which we have just been considering, suggests the following remarks :

If two points, $A$ and $A'$, be so situated with respect to a straight line $CD$, that $CD$ bisects at right angles the straight line joining $A$ and $A'$, then $A$ and $A'$ are said to be *symmetrical* with regard to $CD$.

The importance of symmetrical relations, as suggestive of methods for the solution of problems, cannot be fully shewn to a learner, who is unacquainted with the properties of the circle. The following example, however, will illustrate this part of the subject sufficiently for our purpose at present.

*Find a point in a given straight line, such that the sum of its distances from two fixed points on the same side of the line is a minimum, that is, less than the sum of the distances of any other point in the line from the fixed points.*

Taking the diagram of the last example, suppose $CD$ to be the given line, and $A$, $B$ the given points.

Now if $A$ and $A'$ be symmetrical with respect to $CD$, we know that *every* point in $CD$ is equally distant from $A$ and $A'$. (See Note 8, p. 103.)

Hence the sum of the distances of any point in $CD$ from $A$ and $B$ is equal to the sum of the distances of that point from $A'$ and $B$.

But the sum of the distances of a point in $CD$ from $A'$ and $B$ is the least possible when it lies in the straight line joining $A'$ and $B$.

Hence the point $P$, *determined as in the last example*, is the point required.

NOTE. Propositions IX., X., XI., XII. of Book I. give good examples of symmetrical constructions.

## NOTE 11. *Euclid's Proof of I. 5.*

*The angles at the base of an isosceles triangle are equal to one another; and if the equal sides be produced, the angles upon the other side of the base shall be equal.*

Let $ABC$ be an isosceles △, having $AB = AC$

Produce $AB$, $AC$ to $D$ and $E$.

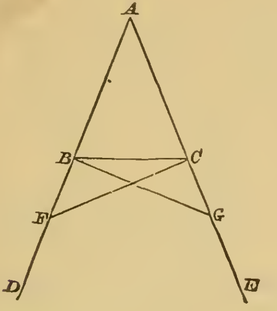
*Then must* $\angle ABC = \angle ACB$,

*and* $\angle DBC = \angle ECB$.

In $BD$ take any pt. $F$.

From $AE$ cut off $AG=AF$.

Join $FC$ and $GB$.



Then in △s $AFC$, $AGB$,

∵ $FA=GA$, and $AC=AB$, and $\angle FAC=\angle GAB$,

∴ $FC=GB$, and $\angle AFC=\angle AGB$, and $\angle ACF=\angle ABG$. I. 4.

Again, ∵ $AF=AG$,

of which the parts $AB$, $AC$ are equal,

∴ remainder $BF$=remainder $CG$. Ax. 3.

Then in △s $BFC$, $CGB$,

∵ $BF=CG$, and $FC=GB$, and $\angle BFC=\angle CGB$.

∴ $\angle FBC=\angle GCB$, and $\angle BCF=\angle CBG$, I. 4.

Now it has been proved that $\angle ACF=\angle ABG$,

of which the parts $\angle BCF$ and $\angle CBG$ are equal;

∴ remaining $\angle ACB$=remaining $\angle ABC$. Ax. 3.

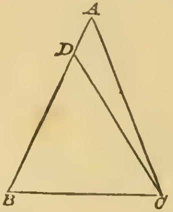
Also it has been proved that $\angle FBC=\angle GCB$,

that is, $\angle DBC=\angle ECB$.

Q. E. D.

## NOTE 12. *Euclid's Proof of I. 6.*

*If two angles of a triangle be equal to one another, the sides also, which subtend the equal angles, shall be equal to one another.*



In $\triangle ABC$ let $\angle ACB = \angle ABC$.

*Then must* $AB = AC$.

For if not, $AB$ is either greater or less than $AC$.

Suppose $AB$ to be greater than $AC$.

From $AB$ cut off $BD = AC$, and join $DC$.

Then in $\triangle$s $DBC$, $ACB$,

$\because DB = AC$, and $BC$ is common, and $\angle DBC = \angle ACB$,

$\therefore \triangle DBC = \triangle ACB$; I. 4.

that is, the less = the greater; which is absurd.

$\therefore AB$ is not greater than $AC$.

Similarly it may be shewn that $AB$ is not less than $AC$;

$\therefore AB = AC$.

Q. E. D.

## NOTE 13. *Euclid's Proof of I. 7.*

*Upon the same base and on the same side of it, there cannot be two triangles that have their sides which are terminated in one extremity of the base equal to one another, and their sides which are terminated in the other extremity of the base equal also.*

If it be possible, on the same base $AB$, and on the same side of it, let there be two $\triangle$s $ACB$, $ADB$, such that $AC = AD$, and also $BC = BD$.

Join $CD$.

First, when the vertex of each of the $\triangle$s is *outside* the other $\triangle$ (Fig. 1.);

FIG 1.

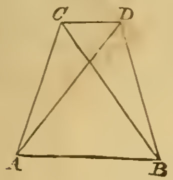
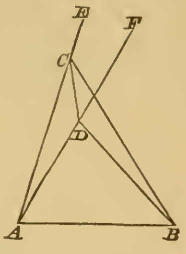


FIG. 2.



$$\because AD=AC,$$

$$\therefore \angle ACD=\angle ADC. \qquad \text{I. 5.}$$

But $\angle ACD$ is greater than $\angle BCD$;

$\therefore \angle ADC$ is greater than $\angle BCD$;

much more is $\angle BDC$ greater than $\angle BCD$.

Again, $\because BC=BD$,

$$\therefore \angle BDC=\angle BCD,$$

that is, $\angle BDC$ is both equal to and greater than $\angle BCD$; which is absurd.

Secondly, when the vertex $D$ of one of the $\triangle$s falls *within* the other $\triangle$ (Fig. 2);

Produce $AC$ and $AD$ to $E$ and $F$

Then $\because AC=AD$.

$$\therefore \angle ECD=\angle FDC. \qquad \text{I. 5.}$$

But $\angle ECD$ is greater than $\angle BCD$;

$\therefore \angle FDC$ is greater than $\angle BCD$;

much more is $\angle BDC$ greater than $\angle BCD$.

Again. $\because BC=BD$,

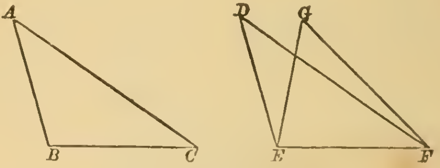
$$\therefore \angle BDC=\angle BCD;$$

that is, $\angle BDC$ is both equal to and greater than $\angle BCD$; which is absurd.

Lastly, when the vertex $D$ of one of the $\triangle$s falls on a side $BC$ of the other, it is plain that $BC$ and $BD$ cannot be equal. Q. E. D.

### NOTE 14. *Euclid's Proof of I. 8.*

*If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise their bases equal, the angle which is contained by the two sides of the one must be equal to the angle contained by the two sides of the other.*



Let the sides of the $\triangle$s $ABC$, $DEF$ be equal, each to each, that is, $AB=DE$, $AC=DF$ and $BC=EF$.

*Then must* $\angle BAC = \angle EDF$.

Apply the $\triangle ABC$ to the $\triangle DEF$.
so that pt. $B$ is on pt. $E$, and $BC$ on $EF$.

Then $\because BC=EF$,
$\therefore C$ will coincide with $F$,
and $BC$ will coincide with $EF$.

Then $AB$ and $AC$ must coincide with $DE$ and $DF$.

For if $AB$ and $AC$ have a different position, as $GE$, $GF$, then upon the same base and upon the same side of it there can be two $\triangle$s, which have their sides which are terminated in one extremity of the base equal, and their sides which are terminated in the other extremity of the base also equal: which is impossible. I. 7.

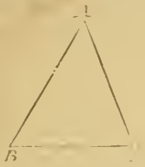
$\therefore$ since base $BC$ coincides with base $EF$,
$AB$ must coincide with $DE$, and $AC$ with $DF$;
$\therefore \angle BAC$ coincides with and is equal to $\angle EDF$.

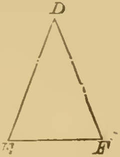
Q. E. D.

### NOTE 15. *Another Proof of I. 24.*

In the $\triangle$s $ABC$, $DEF$, let $AB=DE$ and $AC=DF$, and let $\angle BAC$ be greater than $\angle EDF$.

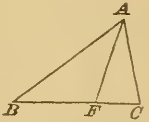
*Then must $BC$ be greater than $EF$.*



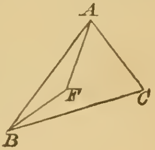


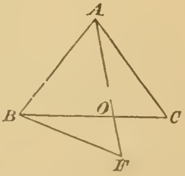
Apply the $\triangle DEF$ to the $\triangle ABC$
so that $DE$ coincides with $AB$.
Then $\because \angle EDF$ is less than $\angle BAC$,
$DF$ will fall between $BA$ and $AC$,
and $F$ will fall *on*, or *above*, or *below*, $BC$.

I. If $F$ fall on $BC$,
$BF$ is less than $BC$;
$\therefore EF$ is less than $BC$.



II. If $F$ fall *above* $BC$,
$BF$, $FA$ together are less than
$BC$, $CA$,
and $FA=CA$;
$\therefore BF$ is less than $BC$;
$\therefore EF$ is less than $BC$.





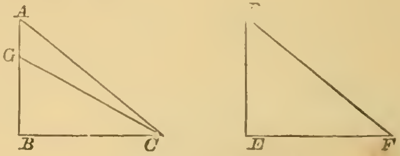
III. If $F$ fall *below* $BC$.

let $AF$ cut $BC$ in $O$.

Then $BO$, $OF$ together are greater than $BF$, I. 20.
and $OC$, $AO$..................................$AC$; I. 20.
$\therefore BC$, $AF$...............................$BF$, $AC$ together,
and $AF=AC$,
$\therefore BC$ is greater than $BF$;
and $\therefore EF$ is less than $BC$. Q. E. D.

### NOTE 16. *Euclid's Proof of I. 26.*

*If two triangles have two angles of the one equal to two angles of the other, each to each, and one side equal to one side, viz., either the sides adjacent to the equal angles, or the sides opposite to equal angles in each; then shall the other sides be equal, each to each; and also the third angle of the one to the third angle of the other.*



In $\triangle$s $ABC$, $DEF$,

Let $\angle ABC = \angle DEF$, and $\angle ACB = \angle DFE$;

and *first*,

Let the sides adjacent to the equal $\angle$s in each be equal, that is, let $BC = EF$.

Then must $AB = DE$, and $AC = DF$, and $\angle BAC = \angle EDF$.

For if $AB$ be not $= DE$, one of them must be the greater.

Let $AB$ be the greater, and make $GB = DE$, and join $GC$.

Then in $\triangle$s $GBC$, $DEF$,

$\because GB = DE$, and $BC = EF$, and $\angle GBC = \angle DEF$,

$\therefore \angle GCB = \angle DFE$. I. 4.

But $\angle ACB = \angle DFE$ by hypothesis;

$\therefore \angle GCB = \angle ACB$;

that is, the less $=$ the greater, which is impossible.

$\therefore AB$ is not greater than $DE$.

In the same way it may be shewn that $AB$ is not less than $DE$;

$\therefore AB = DE$.

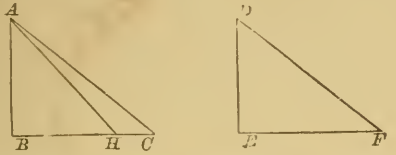
Then in $\triangle$s $ABC$, $DEF$,

$\because AB = DE$, and $BC = EF$, and $\angle ABC = \angle DEF$,

$\therefore AC = DF$, and $\angle BAC = \angle EDF$. I. 4.

*Next*, let the sides which are opposite to equal angles in each triangle be equal, viz., $AB=DE$.

*Then must* $AC=DF$, *and* $BC=EF$, *and* $\angle BAC = \angle EDF$.



For if $BC$ be not$=EF$, let $BC$ be the greater, and make $BH=EF$, and join $AH$.

Then in $\triangle$s $ABH$, $DEF$,

$\because AB=DE$, and $BH=EF$, and $\angle ABH = \angle DEF$,

$\therefore \angle AHB = \angle DFE$. I. 4.

But $\angle ACB = \angle DFE$, by hypothesis,

$\therefore \angle AHB = \angle ACB$;

that is, the exterior $\angle$ of $\triangle AHC$ is equal to the interior and opposite $\angle ACB$, which is impossible.

$\therefore BC$ is not greater than $EF$.

In the same way it may be shewn that $BC$ is not less than $EF$;

$\therefore BC=EF$.

Then in $\triangle$s $ABC$, $DEF$,

$\because AB=DE$, and $BC=EF$, and $\angle ABC = \angle DEF$.

$\therefore AC=DF$, and $\angle BAC = \angle EDF$. I. 4

Q. E. D.

## *Miscellaneous Exercises on Books I. and II.*

1. *AB* and *CD* are equal straight lines, bisecting one another at right angles. Shew that *ACBD* is a square.

2. From a point in the side of a parallelogram draw a line dividing the parallelogram into two equal parts.

3. In the triangle *FDC*, if *FCD* be a right angle, and angle *FDC* be double of angle *CFD*, shew that *FD* is double of *DC*.

4. If *ABC* be an equilateral triangle, and *AD*, *BE* be perpendiculars to the opposite sides intersecting in *F*; shew that the square on *AB* is equal to three times the square on *AF*.

5. Describe a rhombus, which shall be equal to a given triangle, and have each of its sides equal to one side of the triangle.

6. From a given point, outside a given straight line, draw a line making with the given line an angle equal to a given rectilineal angle.

7. If two straight lines be drawn from two given points to meet in a given straight line, shew that the sum of these lines is the least possible, when they make equal angles with the given line.

8. *ABCD* is a parallelogram, whose diagonals *AC*, *BD* intersect in *O*; shew that if the parallelograms *AOBP*, *DOCQ* be completed, the straight line joining *P* and *Q* passes through *O*.

9. *ABCD*, *EBCF* are two parallelograms on the same base *BC*, and so situated that *CF* passes through *A*. Join *DF*, and produce it to meet *BE* produced in *K*; join *FB*, and prove that the triangle *FAB* equals the triangle *FEK*.

10. The alternate sides of a polygon are produced to meet; shew that all the angles at their points of intersection together with four right angles are equal to all the interior angles of the polygon.

11. Shew that the perimeter of a rectangle is always greater than that of the square equal to the rectangle.

12. Shew that the opposite sides of an equiangular hexagon are parallel, though they be not equal.

13. If two equal straight lines intersect each other anywhere at right angles, shew that the area of the quadrilateral formed by joining their extremities is invariable, and equal to one-half the square on either line.

14. Two triangles $ACB$, $ADB$ are constructed on the same side of the same base $AB$. Shew that if $AC=BD$ and $AD=BC$, then $CD$ is parallel to $AB$; but if $AC=BC$ and $AD=BD$, then $CD$ is perpendicular to $AB$.

15. $AB$ is the hypotenuse of a right-angled triangle $ABC$: find a point $D$ in $AB$, such that $DB$ may be equal to the perpendicular from $D$ on $AC$.

16. Find the locus of the vertices of triangles of equal area on the same base, and on the same side of it.

17. Shew that the perimeter of an isosceles triangle is less than that of any triangle of equal area on the same base.

18. If each of the equal angles of an isosceles triangle be equal to one-fourth the vertical angle, and from one of them a perpendicular be drawn to the base, meeting the opposite side produced, then will the part produced, the perpendicular, and the remaining side, form an equilateral triangle.

19. If a straight line terminated by the sides of a triangle be bisected, shew that no other line terminated by the same two sides can be bisected in the same point.

20. Shew how to bisect a given quadrilateral by a straight line drawn from one of its angles.

21. Given the lengths of the two diagonals of a rhombus, construct it.

22. $ABCD$ is a quadrilateral figure: construct a triangle whose base shall be in the line $AB$, such that its altitude shall be equal to a given line, and its area equal to that of the quadrilateral.

23. If from any point in the base of an isosceles triangle perpendiculars be drawn to the sides, their sum will be equal to the perpendicular from either extremity of the base upon the opposite side.

24. If $ABC$ be a triangle, in which $C$ is a right angle, and $DE$ be drawn from a point $D$ in $AC$ at right angles to $AB$, prove that the rectangles $AB$, $AE$ and $AC$, $AD$ are equal.

25. A line is drawn bisecting parallelogram $ABCD$, and meeting $AD$, $BC$ in $E$ and $F$: shew that the triangles $EBF$, $CED$ are equal.

26. Upon the hypotenuse $BC$ and the sides $CA$, $AB$ of a right-angled triangle $ABC$, squares $BDEC$, $AF$ and $AG$ are described: shew that the squares on $DG$ and $EF$ are together equal to five times the square on $BC$.

27. If from the vertical angle of a triangle three straight lines be drawn, one bisecting the angle, the second bisecting the base, and the third perpendicular to the base, shew that the first lies, both in position and magnitude, between the other two.

28. If $ABC$ be a triangle, whose angle $A$ is a right angle, and $BE$, $CF$ be drawn bisecting the opposite sides respectively, shew that four times the sum of the squares on $BE$ and $CF$ is equal to five times the square on $BC$.

29. Let $ACB$, $ADB$ be two right-angled triangles having a common hypotenuse $AB$. Join $CD$ and on $CD$ produced both ways draw perpendiculars $AE$, $BF$. Shew that the sum of the squares on $CE$ and $CF$ is equal to the sum of the squares on $DE$ and $DF$.

30. In the base $AC$ of a triangle take any point $D$: bisect $AD$, $DC$, $AB$, $BC$ at the points $E$, $F$, $G$, $H$ respectively. Shew that $EG$ is equal and parallel to $FH$.

31. If $AD$ be drawn from the vertex of an isosceles triangle $ABC$ to a point $D$ in the base, shew that the rectangle $BD$, $DC$ is equal to the difference between the squares on $AB$ and $AD$.

32. If in the sides of a square four points be taken at equal distances from the four angular points taken in order, the figure contained by the straight lines, which join them, shall also be a square.

33. If the sides of an equilateral and equiangular pentagon be produced to meet, shew that the sum of the angles at the points of meeting is equal to two right angles.

34. Describe a square that shall be equal to the difference between two given and unequal squares.

35. *ABCD*, *AECF* are two parallelograms, *EA*, *AD* being in a straight line. Let *FG*, drawn parallel to *AC*, meet *BA* produced in *G*. Then the triangle *ABE* equals the triangle *ADG*.

36. From *AC*, the diagonal of a square *ABCD*, cut off *AE* equal to one-fourth of *AC*, and join *BE*, *DE*. Shew that the figure *BADE* is equal to twice the square on *AE*.

37. If *ABC* be a triangle, with the angles at *B* and *C* each double of the angle at *A*, prove that the square on *AB* is equal to the square on *BC* together with the rectangle *AB*, *BC*.

38. If two sides of a quadrilateral be parallel, the triangle contained by either of the other sides and the two straight lines drawn from its extremities to the middle point of the opposite side is half the quadrilateral.

39. Describe a parallelogram equal to and equiangular with a given parallelogram, and having a given altitude.

40. If the sides of a triangle taken in order be produced to twice their original lengths, and the outer extremities be joined, the triangle so formed will be seven times the original triangle.

41. If one of the acute angles of a right-angled isosceles triangle be bisected, the opposite side will be divided by the bisecting line into two parts, such that the square on one will be double of the square on the other.

42. *ABC* is a triangle, right-angled at *B*, and *BD* is drawn perpendicular to the base, and is produced to *E* until *ECB* is a right angle; prove that the square on *BC* is equal to the sum of the rectangles *AD*, *DC* and *BD*, *DE*.

43. Shew that the sum of the squares on two unequal lines is greater than twice the rectangle contained by the lines.

44. From a given isosceles triangle cut off a trapezium, having the base of the triangle for one of its parallel sides, and having the other three sides equal.

45. If any number of parallelograms be constructed having their sides of given length, shew that the sum of the squares on the diagonals of each will be the same.

46. $ABCD$ is a right-angled parallelogram, and $AB$ is double of $BC$; on $AB$ an equilateral triangle is constructed: shew that its area will be less than that of the parallelogram.

47. A point $O$ is taken within a triangle $ABC$, such that the angles $BOC$, $COA$, $AOB$ are equal; prove that the squares on $BC$, $CA$, $AB$ are together equal to the rectangles contained by $OB$, $OC$; $OC$, $OA$; $OA$, $OB$; and twice the sum of the squares on $OA$, $OB$, $OC$.

48. If the sides of an equilateral and equiangular hexagon be produced to meet, the angles formed by these lines are together equal to four right angles.

49. $ABC$ is a triangle right-angled at $A$; in the hypotenuse two points $D$, $E$ are taken such that $BD=BA$ and $CE=CA$; shew that the square on $DE$ is equal to twice the rectangle contained by $BE$, $CD$.

50. Given one side of a rectangle which is equal in area to a given square, find the other side.

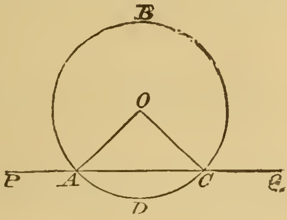
51. $AB$, $AC$ are the two equal sides of an isosceles triangle; from $B$, $BD$ is drawn perpendicular to $AC$, meeting it in $D$; shew that the square on $BD$ is greater than the square on $CD$ by twice the rectangle $AD$, $CD$.

# BOOK III.

## POSTULATE.

A POINT is within, or without, a circle, according as its distance from the centre is less, or greater than the radius of the circle.

DEF. I. A straight line, as $PQ$, drawn so as to cut a circle $ABCD$, is called a SECANT.

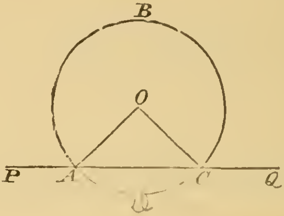


That such a line can only meet the circumference in *two* points may be shewn thus:

Some point within the circle is the centre; let this be $O$. Join $OA$. Then (Ex. 1, I. 16) we can draw one, and only one, straight line from $O$, to meet the straight line $PQ$, such that it shall be equal to $OA$. Let this line be $OC$. Then $A$ and $C$ are the only points in $PQ$, which are *on* the circumference of the circle.

Def. II. The portion $AC$ of the secant $PQ$, intercepted by the circle, is called a Chord.

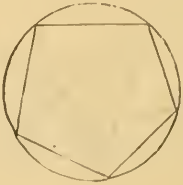
Def. III. The two portions, into which a chord divides the circumference, as $ABC$ and $ADC$, are called Arcs.



Def. IV. The two figures into which a chord divides the circle, as $ABC$ and $ADC$, that is, the figures, of which the boundaries are respectively the arc $ABC$ and the chord $AC$, and the arc $ADC$ and the chord $AC$, are called Segments of the circle.

Def. V. The figure $AOCD$, whose boundaries are two radii and the arc intercepted by them, is called a Sector.

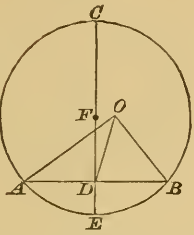
Def. VI. A circle is said to be *described about* a rectilinear figure, when the circumference passes through each of the angular points of the figure.



And the figure is said to be *inscribed* in the circle.

## PROPOSITION I. THEOREM.

*The line, which bisects a chord of a circle at right angles, must contain the centre.*



Let $ABC$ be the given ⊙.

Let the st. line $CE$ bisect the chord $AB$ at rt. angles in $D$.

*Then the centre of the ⊙ must lie in CE.*

For if not, let $O$, a pt. out of $CE$, be the centre ; and join $OA$, $OD$, $OB$.

Then, in △s $ODA$, $ODB$,

$\because AD = BD$, and $DO$ is common, and $OA = OB$ ;

$\therefore \angle ODA = \angle ODB$ ; I. c.

and $\therefore \angle ODB$ is a right $\angle$ . I. Def. 9

But $\angle CDB$ is a right $\angle$ , by construction ;

$\therefore \angle ODB = \angle CDB$, which is impossible ;

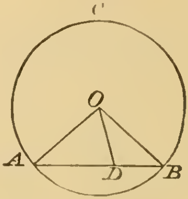
$\therefore O$ is not the centre.

Thus it may be shewn that no point, out of $CE$, can be the centre, and $\therefore$ the centre must lie in $CE$.

COR. *If the chord CE be bisected in F, then F is the centre of the circle.*

## Proposition II. Theorem.

*If any two points be taken in the circumference of a circle, the straight line, which joins them, must fall within the circle.*



Let $A$ and $B$ be any two pts. in the ○ce of the ⊙ $ABC$.

*Then must the st. line $AB$ fall within the* ⊙.

Take any pt. $D$ in the line $AB$.

Find $O$ the centre of the ⊙. III. 1, Cor.

Join $OA$, $OD$, $OB$.

Then $\because \angle OAB = \angle OBA$, I. A.

and $\angle ODB$ is greater than $\angle OAB$, I. 16.

$\therefore \angle ODB$ is greater than $\angle OBA$;

and $\therefore OB$ is greater than $OD$. I. 19.

$\therefore$ the distance of $D$ from $O$ is less than the radius of the ⊙,

and $\therefore D$ lies within the ⊙. Post.

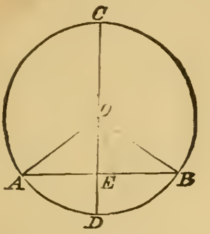
And the same may be shewn of any other pt. in $AB$.

$\therefore AB$ lies entirely within the ⊙.

Q. E. D.

## PROPOSITION III. THEOREM.

*If a straight line, drawn through the centre of a circle, bisect a chord of the circle, which does not pass through the centre, it must cut it at right angles: and conversely, if it cut it at right angles, it must bisect it.*



In the ⊙ $ABC$, let the chord $AB$, which does not pass through the centre $O$, be bisected in $E$ by the diameter $CD$.

*Then must $CD$ be $\perp$ to $AB$.*

Join $OA$, $OB$.

Then in $\triangle$s $AEO$, $BEO$,

$\because AE=BE$, and $EO$ is common, and $OA=OB$,

$\therefore \angle OEA = \angle OEB$. I. c.

Hence $OE$ is $\perp$ to $AB$, I. Def. 9.

that is, $CD$ is $\perp$ to $AB$.

Next let $CD$ be $\perp$ to $AB$.

*Then must $CD$ bisect $AB$.*

For $\because OA=OB$, and $OE$ is common,

in the right-angled $\triangle$s $AEO$, $BEO$,

$\therefore AE=BE$, I. E. Cor. p. 43.

that is, $CD$ bisects $AB$. Q. E. D.

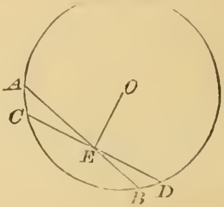
Ex. 1. Shew that, if $CD$ does not cut $AB$ at right angles, it cannot bisect it.

Ex. 2. A line, which bisects two parallel chords in a circle, is also perpendicular to them.

Ex. 3. Through a given point within a circle, which is not the centre, draw a chord which shall be bisected in that point.

### PROPOSITION IV. THEOREM.

*If in a circle two chords, which do not both pass through the centre, cut one another, they do not bisect each other.*



Let the chords $AB$, $CD$, which do not both pass through the centre, cut one another, in the pt. $E$, in the ⊙ $ACBD$.

*Then $AB$, $CD$ do not bisect each other.*

If one of them pass through the centre, it is plainly not bisected by the other, which does not pass through the centre.

But if neither pass through the centre, let, if it be possible, $AE=EB$ and $CE=ED$; find the centre $O$, and join $OE$.

Then ∵ $OE$, passing through the centre, bisects $AB$,

∴ ∠ $OEA$ is a rt. ∠. III. 3.

And ∵ $OE$, passing through the centre, bisects $CD$,

∴ ∠ $OEC$ is a rt. ∠; III. 3

∴ ∠ $OEA$ = ∠ $OEC$, which is impossible;

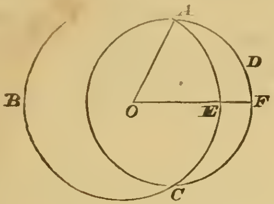
∴ $AB$, $CD$ do not bisect each other. Q. E. D.

Ex. 1 Shew that the locus of the points of bisection of parallel chords of a circle is a straight line.

Ex. 2. Shew that no parallelogram, except those which are rectangular, can be inscribed in a circle.

### Proposition V. Theorem.

*If two circles cut one another, they cannot have the same centre.*



If it be possible, let $O$ be the common centre of the ⊙s $ABC$, $ADC$, which cut one another in the pts. $A$ and $C$.

Join $OA$, and draw $OEF$ meeting the ⊙s in $E$ and $F$.

Then $\because O$ is the centre of ⊙ $ABC$,

$\therefore OE = OA$; I. Def. 13.

and $\because O$ is the centre of ⊙ $ADC$,

$\therefore OF = OA$; I. Def. 13.

$\therefore OE = OF$, which is impossible;

$\therefore O$ is not the common centre.

Q. E. D.

Ex. If two circles cut one another, shew that a line drawn through a point of intersection, terminated by the circumferences and parallel to the line joining the centres, is double of the line joining the centres.

Note. Circles which have the same centre are called *Concentric.*

## NOTE 1. *On the Contact of Circles.*

DEF. VII. Circles are said to touch each other, which meet but do not cut each other.

One circle is said to touch another *internally*, when one point of the circumference of the former lies *on*, and no point *without*, the circumference of the other.

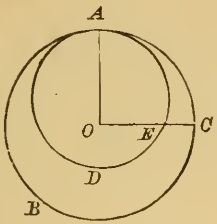
Hence for internal contact one circle must be smaller than the other.

Two circles are said to touch *externally*, when one point of the circumference of the one lies *on*, and no point *within* the circumference of the other.

*N.B.* No restriction is placed by these definitions on the number of points of contact, and it is not till we reach Prop. XIII. that we prove that there can be *but one point of contact.*

## Proposition VI. Theorem.

*If one circle touch another internally, they cannot have the same centre.*



Let ⊙ $ADE$ touch ⊙ $ABC$ internally,
and let $A$ be a point of contact.

Then *some* point $E$ in the ○ce $ADE$ lies *within* ⊙ $ABC$. Def. 7

If it be possible, let $O$ be the common centre of the two ⊙s.

Join $OA$, and draw $OEC$, meeting the ○ces in $E$ and $C$.

Then ∵ $O$ is the the centre of ⊙ $ABC$,
∴ $OA = OC$; I. Def. 13.

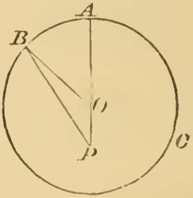
and ∵ $O$ is the centre of ⊙ $ADE$,
∴ $OA = OE$. I. Def. 13.

Hence $OE = OC$, which is impossible,
∴ $O$ is not the common centre of the two ⊙s

Q. E. D.

### PROPOSITION VII. THEOREM.

*If from any point within a circle, which is not the centre, straight lines be drawn to the circumference, the greatest of these lines is that which passes through the centre.*



Let $ABC$ be a ⊙, of which $O$ is the centre.

From $P$, any pt. within the ⊙, draw the st. line $PA$, passing through $O$ and meeting the ○ce in $A$.

*Then must $PA$ be greater than any other st. line, drawn from $P$ to the ○ce.*

For let $PB$ be any other st. line, drawn from $P$ to meet the ○ce in $B$, and join $BO$.

Then $\because AO = BO$,

$\therefore AP =$ sum of $BO$ and $OP$.

But the sum of $BO$ and $OP$ is greater than $BP$, I. 20.

and $\therefore AP$ is greater than $BP$. Q. E. D.

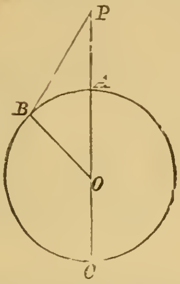
Ex. 1. If $AP$ be produced to meet the circumference in $D$, shew that $PD$ is less than any other straight line that can be drawn from $P$ to the circumference.

Ex. 2. Shew that $PB$ continually decreases, as $B$ passes from $A$ to $D$.

Ex. 3. Shew that two straight lines, but not three, that shall be equal, can be drawn from $P$ to the circumference.

## PROPOSITION VIII. THEOREM.

*If from any point without a circle straight lines be drawn to the circumference, the least of these lines is that which, when produced, passes through the centre, and the greatest is that which passes through the centre.*



Let $ABC$ be a ⊙, of which $O$ is the centre.

From $P$ any pt. outside the ⊙, draw the st. line $PAOC$, meeting the ⊙ce in $A$ and $C$.

*Then must $PA$ be less, and $PC$ greater, than any other st. line drawn from $P$ to the ⊙ce.*

For let $PB$ be any other st. line drawn from $P$ to meet the ⊙ce in $B$, and join $BO$.

Then ∵ sum of $PB$ and $BO$ is greater than $OP$, I. 20.

∴ sum of $PB$ and $BO$ is greater than sum of $AP$ and $AO$.

But $BO=AO$;

∴ $PB$ is greater than $AP$.

Again ∵ $PB$ is less than the sum of $PO$, $OB$, I. 20.

∴ $PB$ is less than the sum of $PO$, $OC$;

∴ $PB$ is less than $PC$. Q. E. D.

Ex. 1. Shew that $PB$ continually increases as $B$ passes from $A$ to $C$.

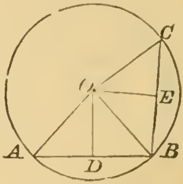
Ex. 2. Shew that from $P$ two straight lines, but not three, that shall be equal, can be drawn to the circumference.

Note. From Props. VII. and VIII. we deduce the following Corollary, which we shall use in the proof of Props. XI. and XIII.

Cor. *If a point be taken, within or without a circle, of all straight lines drawn from it to the circumference, the greatest is that which meets the circumference after passing through the centre,*

### PROPOSITION IX. THEOREM.

*If a point be taken within a circle, from which there fall more than two equal straight lines to the circumference, that point is the centre of the circle.*



Let $O$ be a pt. in the ⊙ $ABC$ from which more than two st. lines $OA$, $OB$, $OC$, drawn to the ○ce, are equal.

*Then must $O$ be the centre of the ⊙.*

Join $AB$, $BC$, and draw $OD$, $OE$ ⊥ to $AB$, $BC$.

Then ∵ $OA = OB$, and $OD$ is common,

in the right-angled △s $AOD$, $BOD$,

∴ $AD = DB$ ; I. E. Cor. p. 43.

∴ the centre of the ⊙ is in $DO$. III. 1.

Similarly it may be shown that

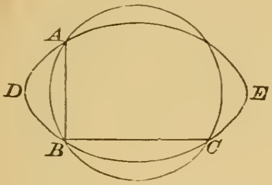
the centre of the ⊙ is $EO$ ;

∴ $O$ is the centre of the ⊙.

Q. E. D

### Proposition X. Theorem.

*Two circles cannot have more than two points common to both, without coinciding entirely.*



If it be possible, let $ABC$ and $ADE$ be two ⊙s which have more than two pts. in common, as $A$, $B$, $C$.

Join $AB$, $BC$.

Then ∵ $AB$ is a chord of each circle,

∴ the centre of each circle lies in the straight line, which bisects $AB$ at right angles; III. 1.

and ∵ $BC$ is a chord of each circle,

∴ the centre of each circle lies in the straight line, which bisects $BC$ at right angles. III. 1.

∴ the centre of each circle is the point, in which the two straight lines, which bisect $AB$ and $BC$ at right angles, meet.

∴ the ⊙s $ABC$, $ADE$ have a common centre, which is impossible; III. 5 and 6.

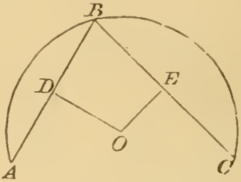
∴ two ⊙s cannot have more than two pts. common to both.

Q. E. D.

Note. We here insert two Propositions, Eucl. III. 25 and IV. 5, which are closely connected with Theorems I. and X. of this book. The learner should compare with this portion of the subject the note on Loci, p. 103.

### PROPOSITION A. PROBLEM. (Eucl. III. 25.)

*An arc of a circle being given, to complete the circle of which it is a part.*



Let $ABC$ be the given arc.

*It is required to complete the ⊙ of which $ABC$ is a part.*

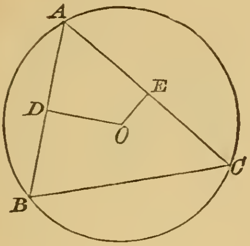
Take $B$, any pt. in arc $ABC$, and join $AB$, $BC$.
From $D$ and $E$, the middle pts. of $AB$ and $BC$,
draw $DO$, $EO$, ⊥s to $AB$, $BC$, meeting in $O$.
Then ∵ $AB$ is to be a chord of the ⊙,
∴ centre of the ⊙ lies in $DO$; III. 1.
and ∵ $BC$ is to be a chord of the ⊙,
∴ centre of the ⊙ lies in $EO$. III. 1.

Hence $O$ is the centre of the ⊙ of which $ABC$ is an arc, and if a ⊙ be described, with centre $O$ and radius $OA$, this will be the ⊙ required.

Q. E. F.

## Proposition B. Problem. (Eucl. iv. 5.)

*To describe a circle about a given triangle.*



Let $ABC$ be the given △.

*It is required to describe a ⊙ about the △.*

From $D$ and $E$, the middle pts. of $AB$ and $AC$, draw $DO$, $EO$, ⊥s to $AB$, $AC$, and let them meet in $O$.

Then ∵ $AB$ is to be a chord of the ⊙,
∴ centre of the ⊙ lies in $DO$. III. 1.
And ∵ $AC$ is to be a chord of the ⊙,
∴ centre of the ⊙ lies in $EO$. III. 1.

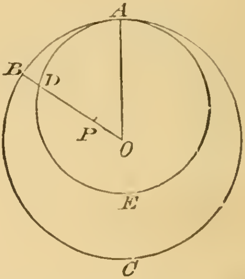
Hence $O$ is the centre of the ⊙ which can be described about the △, and if a ⊙ be described with centre $O$ and radius $OA$, this will be the ⊙ required.

Q. E. F.

Ex. If $BAC$ be a right angle, show that $O$ will coincide with the middle point of $BC$.

### Proposition XI. Theorem.

*If one circle touch another internally at any point, the centre of the interior circle must lie in that radius of the other circle which passes through that point of contact.*



Let the ⊙ $ADE$ touch the ⊙ $ABC$ internally, and let $A$ be a pt. of contact.

Find $O$ the centre of ⊙ $ABC$, and join $OA$.

*Then must the centre of ⊙ $ADE$ lie in the radius $OA$.*

For if not, let $P$ be the centre of ⊙ $ADE$.

Join $OP$, and produce it to meet the ○ces in $D$ and $B$.

Then ∵ $P$ is the centre of ⊙ $ADE$, and from $O$ are drawn to the ○ce of $ADE$ the st. lines $OA$, $OD$, of which $OD$ passes through $P$,

∴ $OD$ is greater than $OA$. III. 8, Cor.

But $OA = OB$;

∴ $OD$ is greater than $OB$,

which is impossible.

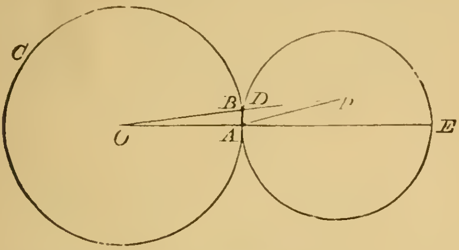
∴ the centre of ⊙ $ADE$ is not out of the radius $OA$.

∴ it lies in $OA$.

Q. E. D.

## Proposition XII. Theorem.

*If two circles touch one another externally at any point, the straight line joining the centre of one with that point of contact must when produced pass through the centre of the other.*



Let ⊙ $ABC$ touch ⊙ $ADE$ externally at the pt. $A$.

Let $O$ be the centre of ⊙ $ABC$.

Join $OA$, and produce it to $E$.

*Then must the centre of ⊙ $ADE$ lie in $AE$.*

For if not, let $P$ be the centre of ⊙ $ADE$.

Join $OP$ meeting the ⊙s in $B$, $D$; and join $AP$.

Then ∵ $OB = OA$,

and $PD = AP$,

∴ $OB$ and $PD$ together $= OA$ and $AP$ together;

∴ $OP$ is not less than $OA$ and $AP$ together.

But $OP$ is less than $OA$ and $AP$ together, I. 20.

which is impossible;

∴ the centre of ⊙ $ADE$ cannot lie out of $AE$.

Q. E. D.

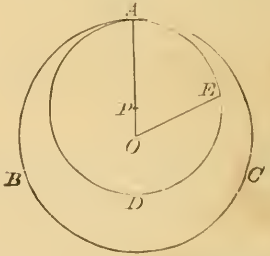
Ex. Three circles touch one another externally, whose centres are $A$, $B$, $C$. Shew that the difference between $AB$ and $AC$ is half as great as the difference between the diameters of the circles, whose centres are $B$ and $C$.

## PROPOSITION XIII. THEOREM.

*One circle cannot touch another at more points than one whether it touch it internally or externally.*

First let the ⊙ *ADE* touch the ⊙ *ABC* *internally* at pt. *A*.

*Then there can be no other point of contact.*



Take *O* the centre of ⊙ *ABC*

Then *P*, the centre of ⊙ *ADE*, lies in *OA*. III. 11.

Take any pt. *E* in the ○ce of the ⊙ *ADE*, and join *OE*.

Then ∵ from *O*, a pt. within or without the ⊙ *ADE*, two lines *OA*, *OE* are drawn to the ○ce, of which *OA* passes through the centre *P*,

∴ *OA* is greater than *OE*, III. 8, Cor.

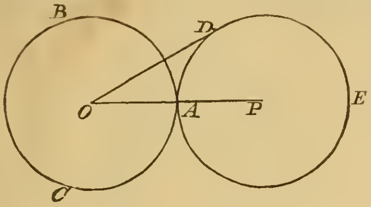
and ∴ *E* is a point *within* the ⊙ *ABC*. Post.

Similarly it may be shewn that every pt. of the ○ce of the ⊙ *ADE*, except *A*, lies *within* the ⊙ *ABC*;

∴ *A* is the only point at which the ⊙s meet.

Next, let the ⊙s *ABC*, *ADE* touch *externally* at the pt. *A*.

*Then there can be no other point of contact.*



Take *O* the centre of the ⊙ *ABC*.

Then *P*, the centre of the ⊙ *ADE*, lies in *OA* produced. III. 12.

Take any pt. *D* in the ◯ce of the ⊙ *ADE*, and join *OD*.

Then ∵ from *O*, a pt. without the ⊙ *ADE*, two lines *OA*, *OD* are drawn to the ◯ce, of which *OA* when produced passes through the centre *P*,

∴ *OD* is greater than *OA*; III. 8.

∴ *D* is a point *without* the ⊙ *ABC*. Post.

Similarly, it may be shewn that every pt. of the ◯ce of *ADE*, except *A*, lies *without* the ⊙ *ABC*;

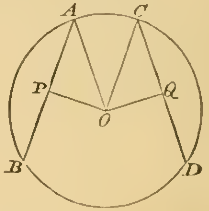
∴ *A* is the only point at which the ⊙s meet.

Q. E. D.

DEF. VIII. The DISTANCE of a chord from the centre is measured by the length of the perpendicular drawn from the centre to the chord.

### Proposition XIV. Theorem.

*Equal chords in a circle are equally distant from the centre; and conversely, those which are equally distant from the centre, are equal to one another.*



Let the chords $AB$, $CD$ in the $\odot$ $ABDC$ be equal.

*Then must* $AB$ *and* $CD$ *be equally distant from the centre* $O$.

Draw $OP$ and $OQ$ $\perp$ to $AB$ and $CD$; and join $AO$, $CO$.
Then $P$ and $Q$ are the middle pts. of $AB$ and $CD$: III. 3.
and $\because AB=CD$, $\therefore AP=CQ$.
Then $\because AP=CQ$, and $AO=CO$,
in the right-angled $\triangle$s $AOP$, $COQ$,
$\therefore OP=OQ$; I. E. Cor. p. 43.
and $\therefore$ $AB$ and $CD$ are equally distant from $O$. Def. 8.

Next, let $AB$ and $CD$ be equally distant from $O$.

*Then must* $AB=CD$.

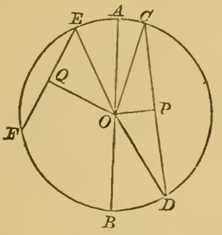
For $\because OP=OQ$, and $AO=CO$,
in the right-angled $\triangle$s $AOP$, $COQ$,
$\therefore AP=CQ$, I. E. Cor.
and $\therefore AB=CD$.

Q. E. D.

Ex. In a circle, whose diameter is 10 inches, a chord is drawn, which is 8 inches long. If another chord be drawn, at a distance of 3 inches from the centre, shew whether it is equal or not to the former.

## PROPOSITION XV. THEOREM.

*The diameter is the greatest chord in a circle, and of all others that which is nearer to the centre is always greater than one more remote; and the greater is nearer to the centre than the less.*



Let $AB$ be a diameter of the ⊙ $ABDC$, whose centre is $O$, and let $CD$ be any other chord, not a diameter, in the ⊙, nearer to the centre than the chord $EF$.

*Then must AB be greater than CD, and CD greater than EF.*

Draw $OP$, $OQ$ $\perp$ to $CD$ and $EF$; and join $OC$, $OD$, $OE$.

Then $\because AO=CO$, and $OB=OD$, I. Def. 13.

$\therefore AB=$sum of $CO$ and $OD$,

and $\therefore AB$ is greater than $CD$. I. 20.

Again, $\because CD$ is nearer to the centre than $EF$,

$\therefore OP$ is less than $OQ$. Def. 8.

Now $\because$ sq. on $OC=$sq. on $OE$,

$\therefore$ sum of sqq. on $OP$, $PC=$sum of sqq on $OQ$, $QE$. I. 47.

But sq. on $OP$ is less than sq. on $OQ$;

$\therefore$ sq. on $PC$ is greater than sq. on $QE$;

$\therefore PC$ is greater than $QE$;

and $\therefore CD$ is greater than $EF$.

Next, let $CD$ be greater than $EF$.

*Then must CD be nearer to the centre than EF.*

For $\because CD$ is greater than $EF$,

$\therefore PC$ is greater than $QE$.

Now the sum of sqq. on $OP$, $PC$ = sum of sqq. on $OQ$, $QE$.
But sq. on $PC$ is greater than sq. on $QE$;
∴ sq. on $OP$ is less than sq. on $OQ$;
∴ $OP$ is less than $OQ$;
and ∴ $CD$ is nearer to the centre than $EF$.

Q. E. D.

Ex. 1. Draw a chord of given length in a given circle, which shall be bisected by a given chord.

Ex. 2. If two isosceles triangles be of equal altitude, and the sides of one be equal to the sides of the other, shew that their bases must be equal.

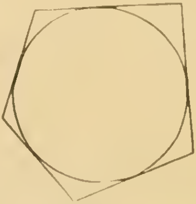
Ex. 3. Any two chords of a circle, which cut a diameter in the same point and at equal angles, are equal to one another.

Def. IX. *A straight line is said to be a* Tangent *to, or to touch, a circle, when it meets and, being produced, does not cut the circle.*

From this definition it follows that the tangent meets the circle in one point only, for if it met the circle in two points it would cut the circle, since the line joining two points in the circumference is, being produced, a secant. (III. 2.)

Def. X. If from any point in a circle a line be drawn at right angles to the tangent at that point, the line is called a Normal to the circle at that point.

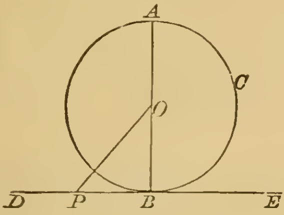
Def. XI. A rectilinear figure is said to be *described about* a circle, when each side of the figure touches the circle.



And the circle is said to be *inscribed* in the figure.

## PROPOSITION XVI. THEOREM.

*The straight line drawn at right angles to the diameter of a circle, from the extremity of it, is a tangent to the circle.*



Let $ABC$ be a ⊙, of which the centre is $O$, and the diameter $AOB$

Through $B$ draw $DE$ at right angles to $AOB$. I. 11.

*Then must $DE$ be a tangent to the ⊙.*

Take any point $P$ in $DE$, and join $OP$.

Then, ∵ ∠ $OBP$ is a right angle,

∴ ∠ $OPB$ is less than a right angle, I. 17.

and ∴ $OP$ is greater than $OB$. I. 19.

Hence $P$ is a point without the ⊙ $ABC$. P

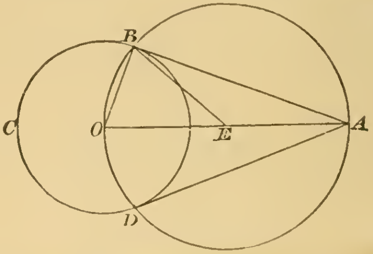
In the same way it may be shewn that every point in $D$ or $DE$ produced in either direction, except the point $B$, lies without the ⊙;

∴ $DE$ is a tangent to the ⊙. Def. 9.

Q. E. D.

### PROPOSITION XVII. PROBLEM.

*To draw a straight line from a given point, either* WITHOUT *or* ON *the circumference, which shall touch a given circle.*



Let $A$ be the given pt., *without* the ⊙ $BCD$.

Take $O$ the centre of ⊙ $BCD$, and join $OA$.

Bisect $OA$ in $E$, and with centre $E$ and radius $EO$ describe ⊙ $ABOD$, cutting the given ⊙ in $B$ and $D$.

*Join* $AB$, $AD$. *These are tangents to the* ⊙ $BCD$.

Join $BO$, $BE$.

Then ∵ $OE=BE$, ∴ $\angle OBE=\angle BOE$; I. A.

∴ $\angle AEB=$ twice $\angle OBE$; I. 32.

and ∵ $AE=BE$, ∴ $\angle ABE=\angle BAE$; I. A.

∴ $\angle OEB=$ twice $\angle ABE$; I. 32.

∴ sum of ∠s $AEB$, $OEB=$ twice sum of ∠s $OBE$, $ABE$, that is, two right angles $=$ twice $\angle OBA$;

∴ $\angle OBA$ is a right angle,

and ∴ $AB$ is a tangent to the ⊙ $BCD$. III. 16.

Similarly it may be shewn that $AD$ is a tangent to ⊙ $BCD$.

Next, let the given pt. be *on* the ○ce of the ⊙, as $B$.

Then, if $BA$ be drawn ⊥ to the radius $OB$,

$BA$ is a tangent to the ⊙ at $B$. III. 16.

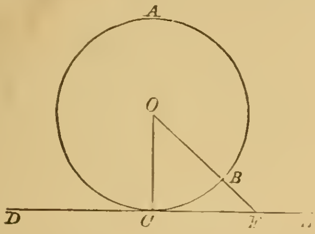
Q. E. D.

Ex. 1. Shew that the two tangents, drawn from a point without the circumference to a circle, are equal.

Ex. 2. If a quadrilateral $ABCD$ be described about a circle, shew that the sum of $AB$ and $CD$ is equal to the sum of $AD$ and $BC$.

## PROPOSITION XVIII. THEOREM.

*If a straight line touch a circle, the straight line drawn from the centre to the point of contact must be perpendicular to the line touching the circle.*



Let the st. line $DE$ touch the ⊙ $ABC$ in the pt. $C$.

Find $O$ the centre, and join $OC$.

*Then must $OC$ be $\perp$ to $DE$.*

For if it be not, draw $OBF \perp$ to $DE$, meeting the ○ce in $B$.

Then $\because \angle OFC$ is a rt. angle,

$\therefore \angle OCF$ is less than a rt. angle, I. 17.

and $\therefore OC$ is greater than $OF$. I. 19.

But $OC = OB$,

$\therefore OB$ is greater than $OF$, which is impossible;

$\therefore OF$ is not $\perp$ to $DE$, and in the same way it may be shewn that no other line drawn from $O$, but $OC$, is $\perp$ to $DE$,

$\therefore OC$ is $\perp$ to $DE$.

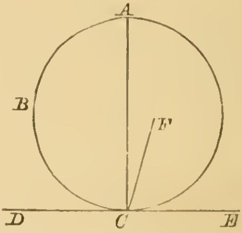
Q. E. D.

Ex. If two straight lines intersect, the centres of all circles touched by both lines lie in two lines at right angles to each other.

NOTE. Prop. XVIII. might be stated thus:—*All radii of a circle are normals to the circle at the points where they meet the circumference.*

### PROPOSITION XIX. THEOREM.

*If a straight line touch a circle, and from the point of contact a straight line be drawn at right angles to the touching line, the centre of the circle must be in that line.*



Let the st. line $DE$ touch the ⊙ $ABC$ at the pt. $C$, and from $C$ let $CA$ be drawn ⊥ to $DE$.

*Then must the centre of the ⊙ be in $CA$.*

For if not, let $F$ be the centre, and join $FC$.

Then ∵ $DCE$ touches the ⊙, and $FC$ is drawn from centre to pt. of contact,

$\therefore \angle FCE$ is a rt. angle. III. 18.

But $\angle ACE$ is a rt. angle.

$\therefore \angle FCE = \angle ACE$, which is impossible.

In the same way it may be shewn that no pt. out of $CA$ can be the centre of the ⊙;

∴ the centre of the ⊙ lies in $CA$.

Q. E. D.

Ex. Two concentric circles being described, if a chord of the greater touch the less, the parts of the chord, intercepted between the two circles, are equal.

NOTE. Prop. XIX. might be stated thus:—*Every normal to a circle passes through the centre.*

### PROPOSITION XX. THEOREM.

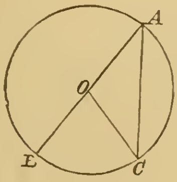
*The angle at the centre of a circle is double of the angle at the circumference, subtended by the same arc.*

Let $ABC$ be a ⊙, $O$ the centre,

$BC$ any arc, $A$ any pt. in the ○ce.

*Then must* $\angle BOC =$ *twice* $\angle BAC$.

First, suppose $O$ to be in one of the lines containing the $\angle BAC$.



Then $\because OA = OC$,

$\therefore \angle OCA = \angle OAC$; I. A.

$\therefore$ sum of $\angle$s $OCA$, $OAC =$ twice $\angle OAC$.

But $\angle BOC =$ sum of $\angle$s $OCA$, $OAC$, I. 32.

$\therefore \angle BOC =$ twice $\angle OAC$.

that is, $\angle BOC =$ twice $\angle BAC$

Next, suppose $O$ to be within (fig 1), or without (fig. 2) the $\angle BAC$.

Fig. 1.

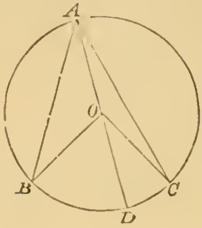
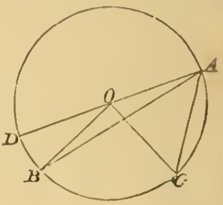


Fig. 2.



Join $AO$, and produce it to meet the $\odot$ce in $D$.

Then, as in the first case,

$$\angle COD = \text{twice } \angle CAD,$$

$$\text{and } \angle BOD = \text{twice } \angle BAD;$$

$\therefore$, fig. 1, sum of $\angle$s $COD$, $BOD$ = twice sum of $\angle$s $CAD$, $BAD$,

$$\text{that is, } \angle BOC = \text{twice } \angle BAC.$$

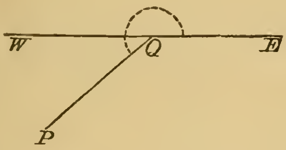
And, fig. 2, difference of $\angle$s $COD$, $BOD$ = twice difference of $\angle$s $CAD$, $BAD$, that is, $\angle BOC$ = twice $\angle BAC$.

Q. E. D.

Ex. From any point in a straight line, touching a circle, a straight line is drawn through the centre, and is terminated by the circumference; the angle between these two straight lines is bisected by a straight line, which intersects the straight line joining their extremities. Shew that the angle between the last two lines is half a right angle.

## NOTE 2. *On Flat and Reflex Angles.*

We have already explained (Note 3, Book I., p. 28) how Euclid's definition of an angle may be extended with advantage, so as to include the conception of an angle equal to two right angles: and we now proceed to shew how the Definition given in that Note may be extended, so as to embrace angles greater than two right angles.



Let $WQ$ be a straight line, and $QE$ its continuation.

Then, by the Definition, the angle made by $WQ$ and $QE$, which we propose to call a FLAT ANGLE, is equal to two right angles.

Now suppose $QP$ to be a straight line, which revolves about the fixed point $Q$, and which at first coincides with $QE$.

When $QP$, revolving from right to left, coincides with $QW$, it has described an angle equal to two right angles.

When $QP$ has continued its revolution, so as to come into the position indicated in the diagram, it has described an angle $EQP$, indicated by the dotted line, greater than two right angles, and this we call a REFLEX ANGLE.

To assist the learner, we shall mark these angles with dotted lines in the diagrams.

Admitting the existence of angles, equal to and greater than two right angles, the Proposition last proved may be extended, as we now proceed to shew,

### PROPOSITION C. THEOREM.

*The angle, not less than two right angles, at the centre of a circle is double of the angle at the circumference, subtended by the same arc.*

Fig. 1.

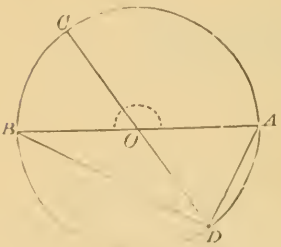
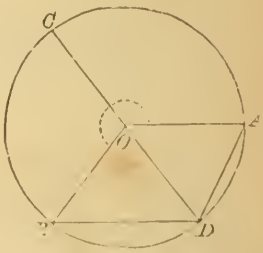


Fig. 2.



In the ⊙ $ACBD$, let the angles $AOB$ (not less than two right angles) at the centre, and $ADB$ at the circumference, be subtended by the same arc $ACB$.

*Then must* $\angle AOB =$ *twice* $\angle ADB$.

Join $DO$, and produce it to meet the arc $ACB$ in $C$.

Then $\because \angle AOC =$ twice $\angle ADO$, III. 20.

and $\angle BOC =$ twice $\angle BDO$, III. 20.

$\therefore$ sum of $\angle$ s $AOC$, $BOC =$ twice sum of $\angle$ s $ADO$, $BDO$,

that is, $\angle AOB =$ twice $\angle ADB$.

Q. E. D.

NOTE. In fig. 1, $\angle AOB$ is drawn a flat angle,

and in fig. 2, $\angle AOB$ is drawn a reflex angle.

DEF. XII. The angle in a segment is the angle contained by two straight lines drawn from any point in the arc to the extremities of the chord.

## PROPOSITION XXI. THEOREM.

*The angles in the same segment of a circle are equal to one another.*

Fig. 1.

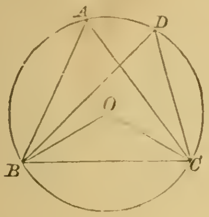
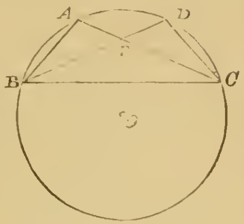


Fig. 2.



Let $BAC$, $BDC$ be angles in the same segment $BADC$.

*Then must* $\angle BAC = \angle BDC$.

First, when segment $BADC$ is greater than a semicircle,

From $O$, the centre, draw $OB$, $OC$. (Fig. 1.)

Then, ∵ $\angle BOC =$ twice $\angle BAC$, III. 20.

and $\angle BOC =$ twice $\angle BDC$, III. 20.

∴ $\angle BAC = \angle BDC$.

Next, when segment $BADC$ is less than a semicircle.

Let $E$ be the pt. of intersection of $AC$, $DB$. (Fig. 2.)

Then ∵ $\angle ABE = \angle DCE$, by the first case,

and $\angle BEA = \angle CED$, I. 15.

∴ $\angle EAB = \angle EDC$, I. 32.

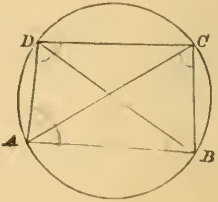
that is, $\angle BAC = \angle BDC$. Q. E. D.

Ex. 1. Shew that, by assuming the possibility of an angle being greater than two right angles, both the cases of this proposition may be included in one.

Ex. 2. If two straight lines, whose extremities are in the circumference of a circle, cut one another, the triangles formed by joining their extremities are equiangular to each other.

## PROPOSITION XXII. THEOREM.

*The opposite angles of any quadrilateral figure, inscribed in a circle, are together equal to two right angles.*



Let $ABCD$ be a quadrilateral fig. inscribed in a $\odot$.

*Then must each pair of its opposite* $\angle$*s be together equal to two rt.* $\angle$*s.*

Draw the diagonals $AC$, $BD$.

Then $\because \angle ADB = \angle ACB$, in the same segment, III. 21.

and $\angle BDC = \angle BAC$, in the same segment; III. 21.

$\therefore$ sum of $\angle$s $ADB$, $BDC$ = sum of $\angle$s $ACB$, $BAC$;

that is, $\angle ADC$ = sum of $\angle$s $ACB$, $BAC$.

Add to each $\angle ABC$.

Then $\angle$s $ADC$, $ABC$ together = sum of $\angle$s $ACB$, $BAC$, $ABC$;

and $\therefore \angle$s $ADC$, $ABC$ together = two right $\angle$s. I. .

Similarly, it may be shewn,

that $\angle$s $BAD$, $BCD$ together = two right $\angle$s.

Q. E. D.

NOTE.—Another method of proving this proposition is given on page 177.

Ex. 1. If one side of a quadrilateral figure inscribed in a circle be produced, the exterior angle is equal to the opposite angle of the quadrilateral.

Ex. 2. If the sides *AB*, *DC* of a quadrilateral inscribed in a circle be produced to meet in *E*, then the triangles *EBC*, *EAD* will be equiangular.

Ex. 3. Shew that a circle cannot be described about a rhombus.

Ex. 4. The lines, bisecting any angle of a quadrilateral figure inscribed in a circle and the opposite exterior angle, meet in the circumference of the circle.

Ex. 5. *AB*, a chord of a circle, is the base of an isosceles triangle, whose vertex *C* is without the circle, and whose equal sides meet the circle in *D*, *E*: shew that *CD* is equal to *CE*.

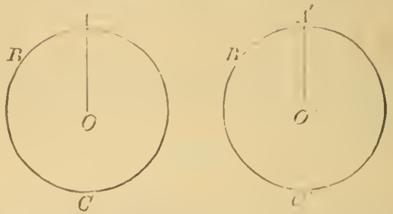
Ex. 6. If in any quadrilateral the opposite angles be together equal to two right angles, a circle may be described about that quadrilateral.

Propositions XXIII. and XXIV., not being required in the method adopted for proving the subsequent Propositions in this book, are removed to the Appendix. Proposition XXV. has been already proved.

### NOTE 3. *On the Method of Superposition, as applied to Circles.*

In Props. XXVI. XXVII. XXVIII. XXIX. we prove certain relations existing between chords, arcs, and angles in equal circles. As we shall employ the Method of Superposition, we must state the principles which render this method applicable, as a test of equality, in the case of figures with *circular* boundaries.

DEF. XIII. *Equal circles are those, of which the radii are equal.*



For suppose $ABC$, $A'B'C'$ to be circles, of which the radii are equal.

Then if ⊙ $A'B'C'$ be applied to ⊙ $ABC$, so that $O'$, the centre of $A'B'C'$, coincides with $O$, the centre of $ABC$, it is evident that any *particular* point $A'$ in the ○ce of the former must coincide with *some* point $A$ in ○ce of the latter, because of the equality of the radii $O'A'$ and $OA$.

Hence ○ce $A'B'C'$ must coincide with ○ce $ABC$,
that is, ⊙ $A'B'C'$ = ⊙ $ABC$.

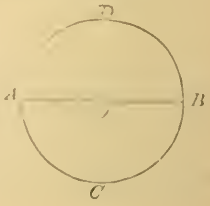
Further, when we have applied the circle $A'B'C'$ to the circle $ABC$, so that the centres coincide, we may imagine $ABC$ to remain fixed, while $A'B'C'$ revolves round the common centre. Hence we may suppose any particular point $B'$ in the circumference of $A'B'C'$ to be made to coincide with any particular point $B$ in the circumference of $ABC$.

Again, any radius $O'A'$ of the circle $A'B'C'$ may be made to coincide with any radius $OA$ of the circle $ABC$.

Also, if $A'B'$ and $AB$ be equal arcs, they may be made to coincide.

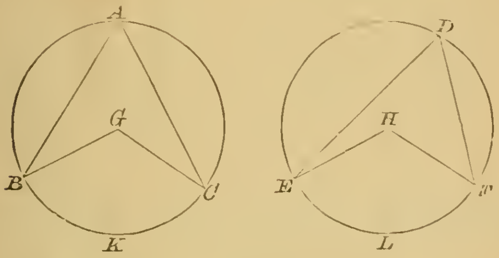
Again, every diameter of a circle divides the circle into equal segments.

For let $AOB$ be a diameter of the circle $ACBD$, of which $O$ is the centre. Suppose the segment $ACB$ to be applied to the segment $ADB$, so as to keep $AB$ a common boundary: then the arc $ACB$ must coincide with the arc $ADB$, because every point in each is equally distant from $O$.



## PROPOSITION XXVI. THEOREM.

*In equal circles, the arcs, which subtend equal angles, whether they be at the centres or at the circumferences, must be equal.*



Let $ABC$, $DEF$ be equal circles, and let $\angle$s $BGC$, $EHF$ at their centres, and $\angle$s $BAC$, $EDF$ at their ○ces, be equal.

*Then must arc $BKC$ = arc $ELF$.*

For, if ⊙ $ABC$ be applied to ⊙ $DEF$,

so that $G$ coincides with $H$, and $GB$ falls on $HE$,

then, $\because GB = HE$, $\therefore B$ will coincide with $E$.

And $\because \angle BGC = \angle EHF$, $\therefore GC$ will fall on $HF$;

and $\because GC = HF$, $\therefore C$ will coincide with $F$.

Then $\because B$ coincides with $E$ and $C$ with $F$,

$\therefore$ arc $BKC$ will coincide with and be equal to arc $ELF$.

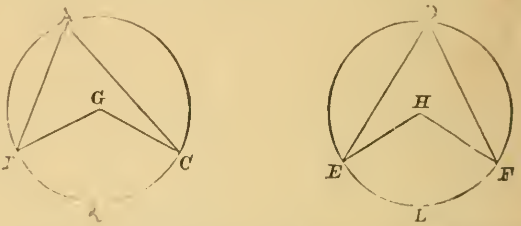
Q. E. D.

COR. Sector $BGCK$ is equal to sector $EHFL$.

NOTE. This and the three following Propositions are, and will hereafter be assumed to be, true for *the same circle* as well as for *equal circles.*

### PROPOSITION XXVII. THEOREM.

*In equal circles, the angles, which are subtended by equal arcs, whether they are at the centres or at the circumferences, must be equal.*



Let $ABC$, $DEF$ be equal circles, and let $\angle$s $BGC$, $EHF$ at their centres, and $\angle$s $BAC$, $EDF$ at their ○ces, be subtended by equal arcs $BKC$, $ELF$.

*Then must* $\angle BGC = \angle EHF$, *and* $\angle BAC = \angle EDF$.

For, if ⊙ $ABC$ be applied to ⊙ $DEF$,
so that $G$ coincides with $H$, and $GB$ falls on $HE$,
then $\because GB = HE$, $\therefore B$ will coincide with $E$;
and $\because$ arc $BKC$ = arc $ELF$, $\therefore C$ will coincide with $F$.
Hence, $GC$ will coincide with $HF$.
Then $\because BG$ coincides with $EH$, and $GC$ with $HF$,
$\therefore \angle BGC$ will coincide with and be equal to $\angle EHF$.
Again, $\because \angle BAC$ = half of $\angle BGC$, III. 20.
and $\angle EDF$ = half of $\angle EHF$, III. 20.
$\therefore \angle BAC = \angle EDF$. I. Ax. 7.

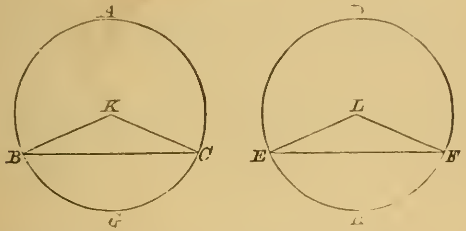
Q. E. D.

Ex. 1. If, in a circle, $AB$, $CD$ be two arcs of given magnitude, and $AC$, $BD$ be joined to meet in $E$, shew that the angle $AEB$ is invariable.

Ex. 2. The straight lines joining the extremities of the chords of two equal arcs of the same circle, towards the same parts, are parallel to each other.

### PROPOSITION XXVIII. THEOREM.

*In equal circles, the arcs, which are subtended by equal chords, must be equal, the greater to the greater, and the less to the less.*



Let $ABC$, $DEF$ be equal circles, and $BC$, $EF$ equal chords, subtending the major arcs $BAC$, $EDF$,

and the minor arcs $BGC$, $EHF$.

*Then must arc* $BAC$ = *arc* $EDF$, *and arc* $BGC$ = *arc* $EHF$.

Take the centres $K$, $L$, and join $KB$, $KC$, $LE$, $LF$.

Then $\because KB=LE$, and $KC=LF$, and $BC=EF$,

$\therefore \angle BKC = \angle ELF$. I. c.

Hence, if $\odot ABC$ be applied to $\odot DEF$,

so that $K$ coincides with $L$, and $KB$ falls on $LE$,

then $\because \angle BKC = \angle ELF$, $\therefore KC$ will fall on $LF$;

and $\because KC = LF$, $\therefore C$ will coincide with $F$.

Then $\because B$ coincides with $E$, and $C$ with $F$,

$\therefore$ arc $BAC$ will coincide with and be equal to arc $EDF$,

and arc $BGC$..................................................$EHF$.

Q. E. D.

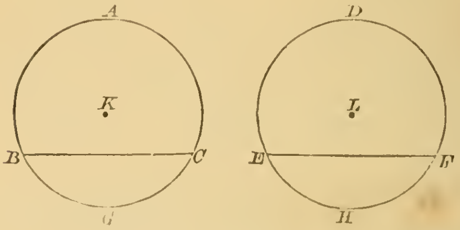
Ex. 1. If, in a circle $ABCD$, the chord $AB$ be equal to the chord $DC$, $AD$ must be parallel to $BC$.

Ex. 2. If a straight line, drawn from $A$ the middle point of an arc $BC$, touch the circle, shew that it is parallel to the chord $BC$.

Ex. 3. If two equal chords, in a given circle, cut one another, the segments of the one shall be equal to the segments of the other, each to each.

### PROPOSITION XXIX. THEOREM.

*In equal circles, the chords, which subtend equal arcs, must be equal.*



Let $ABC$, $DEF$ be equal circles, and let $BC$, $EF$ be chords subtending the equal arcs $BGC$, $EHF$.

*Then must chord* $BC$ = *chord* $EF$.

Take the centres $K$, $L$.

Then, if ⊙ $ABC$ be applied to ⊙ $DEF$,

so that $K$ coincides with $L$, and $B$ with $E$,

and arc $BGC$ falls on arc $EHF$,

$\because$ arc $BGC$ = arc $EHF$, $\therefore$ $C$ will coincide with $F$.

Then $\because$ $B$ coincides with $E$ and $C$ with $F$,

$\therefore$ chord $BC$ must coincide with and be equal to chord $EF$.

Q. E. D.

Ex. 1. The two straight lines in a circle, which join the extremities of two parallel chords, are equal to one another.

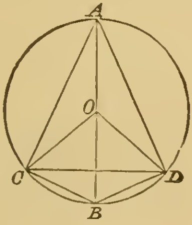
Ex. 2. If three equal chords of a circle, cut one another in the same point, within the circle, that point is the centre.

## NOTE 4. *On the Symmetrical properties of the Circle with regard to its diameter.*

The brief remarks on Symmetry in pp. 107, 108 may now be extended in the following way:

A *figure* is said to be symmetrical with regard to a line, when every perpendicular to the line meets the figure at points which are equidistant from the line.

Hence a Circle is Symmetrical with regard to its Diameter, because the diameter *bisects* every chord, to which it is perpendicular.



Further, suppose $AB$ to be a diameter of the circle $ACBD$, of which $O$ is the centre, and $CD$ to be a chord perpendicular to $AB$.

Then, if lines be drawn as in the diagram, we know that $AB$ bisects

(1.) The chord $CD$, III. 1.

(2.) The arcs $CAD$ and $CBD$, III. 26.

(3.) The angles $CAD$, $COD$, $CBD$, and the reflex angle $DOC$. I. 4.

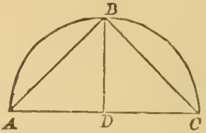
Also, chord $CB$ = chord $DB$, I. 4.

and chord $AC$ = chord $AD$. I. 4.

These Symmetrical relations should be carefully observed, because they are often suggestive of methods for the solution of problems.

### PROPOSITION XXX. PROBLEM.

*To bisect a given arc.*



Let $ABC$ be the given arc.

*It is required to bisect the arc $ABC$.*

Join $AC$, and bisect the chord $AC$ in $D$. I. 10.

From $D$ draw $DB \perp$ to $AC$. I. 11.

*Then will the arc $ABC$ be bisected in $B$.*

Join $BA$, $BC$.

Then, in $\triangle$s $ADB$, $CDB$,

$\because AD = CD$, and $DB$ is common, and $\angle ADB = \angle CDB$,

$\therefore BA = BC$. I. 4.

But, in the same circle, the arcs, which are subtended by equal chords, are equal, the greater to the greater and the less to the less; III. 28.

and $\because BD$, if produced, is a diameter,

$\therefore$ each of the arcs $BA$, $BC$, is less than a semicircle,

and $\therefore$ arc $BA$ = arc $BC$.

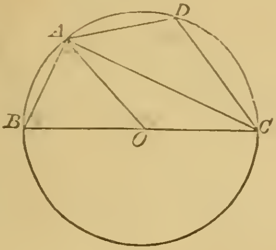
Thus the arc $ABC$ is bisected in $B$.

Q. E. F.

Ex. If, from any point in the diameter of a semicircle, there be drawn two straight lines to the circumference, one to the bisection of the circumference, and the other at right angles to the diameter, the squares on these two lines are together double of the square on the radius.

## PROPOSITION XXXI. THEOREM.

*In a circle, the angle in a semicircle is a right angle: and the angle in a segment greater than a semicircle is less than a right angle; and the angle in a segment less than a semicircle is greater than a right angle.*



Let $ABC$ be a ⊙, $O$ its centre, and $BC$ a diameter.

Draw $AC$, dividing the ⊙ into the segments $ABC$, $ADC$.

Join $BA$, $AD$, $DC$, $AO$.

*Then must the ∠ in the semicircle $BAC$ be a rt. ∠, and ∠ in segment $ABC$, greater than a semicircle, less than a rt. ∠, and ∠ in segment $ADC$, less than a semicircle, greater than a rt. ∠.*

First, ∵ $BO=AO$, ∴ ∠ $BAO=$ ∠ $ABO$; I. A.

∴ ∠ $COA=$twice ∠ $BAO$; I. 32.

and ∵ $CO=AO$, ∴ ∠ $CAO=$ ∠ $ACO$; I. A.

∴ ∠ $BOA=$twice ∠ $CAO$; I. 32.

∴ sum of ∠ s $COA$, $BOA=$twice sum of ∠ s $BAO$, $CAO$, that is, two right angles=twice ∠ $BAC$.

∴ ∠ $BAC$ is a right angle.

Next, ∵ ∠ $BAC$ is a rt. ∠,

∴ ∠ $ABC$ is less than a rt. ∠. I. 17.

Lastly, ∵ sum of ∠ s $ABC$, $ADC=$two rt. ∠ s, III. 22.

and ∠ $ABC$ is less than a rt. ∠,

∴ ∠ $ADC$ is greater than a rt. ∠. Q. E. D.

NOTE.—For a simpler proof see page 178.

Ex. 1. If a circle be described on the radius of another circle as diameter, any straight line, drawn from the point, where they meet, to the outer circumference, is bisected by the interior one

Ex. 2. If a straight line be drawn to touch a circle, and be parallel to a chord, the point of contact will be the middle point of the arc cut off by the chord.

Ex. 3. If, from any point without a circle, lines be drawn touching it, the angle contained by the tangents is double of the angle contained by the line joining the points of contact, and the diameter drawn through one of them.

Ex. 4. The vertical angle of any oblique-angled triangle inscribed in a circle is greater or less than a right angle, by the angle contained by the base and the diameter drawn from the extremity of the base.

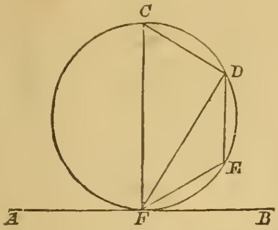
Ex. 5. If, from the extremities of any diameter of a given circle, perpendiculars be drawn to any chord of the circle that is not parallel to the diameter, the less perpendicular shall be equal to that segment of the greater, which is contained between the circumference and the chord.

Ex. 6. If two circles cut one another, and from either point of intersection diameters be drawn, the extremities of these diameters and the other point of intersection lie in the same straight line.

Ex. 7. Draw a straight line cutting two concentric circles, so that the part of it which is intercepted by the circumference of the greater may be twice the part intercepted by the circumference of the less.

### PROPOSITION XXXII. THEOREM.

*If a straight line touch a circle, and from the point of contact a straight line be drawn cutting the circle, the angles made by this line with the line touching the circle must be equal to the angles, which are in the alternate segments of the circle.*



Let the st. line $AB$ touch the ⊙ $CDEF$ in $F$.
Draw the chord $FD$, dividing the ⊙ into segments $FCD$, $FED$.
*Then must* $\angle DFB = \angle$ *in segment* $FCD$,
*and* $\angle DFA = \angle$ *in segment* $FED$.

From $F$ draw the chord $FC \perp$ to $AB$.
Then $FC$ is a diameter of the ⊙. III. 19.
Take any pt. $E$ in the arc $FED$, and join $FE$, $ED$, $DC$.
Then $\because FDC$ is a semicircle, $\therefore \angle FDC$ is a rt. $\angle$; III. 31.
$\therefore$ sum of $\angle$ s $FCD$, $CFD =$ a rt. $\angle$. I. 32.
Also, sum of $\angle$ s $DFB$, $CFD =$ a rt. $\angle$.
$\therefore$ sum of $\angle$ s $DFB$, $CFD =$ sum of $\angle$ s $FCD$, $CFD$,
and $\therefore \angle DFB = \angle FCD$,
that is, $\angle DFB = \angle$ in segment $FCD$.

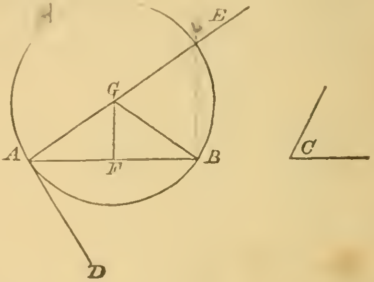
Again, $\because CDEF$ is a quadrilateral fig. inscribed in a ⊙,
$\therefore$ sum of $\angle$ s $FED$, $FCD =$ two rt. $\angle$ s. III. 22.
Also, sum of $\angle$ s $DFA$, $DFB =$ two rt. $\angle$ s. I. 13.
$\therefore$ sum of $\angle$ s $DFA$, $DFB =$ sum of $\angle$ s $FED$, $FCD$;
and $\angle DFB$ has been proved $= \angle FCD$;
$\therefore \angle DFA = \angle FED$,
that is, $\angle DFA = \angle$ in segment $FED$.

Q. E. D.

Ex. The chord joining the points of contact of parallel tangents is a diameter.

### PROPOSITION XXXIII. PROBLEM.

*On a given straight line to describe a segment of a circle capable of containing an angle equal to a given angle.*



Let $AB$ be the given st. line, and $C$ the given $\angle$.

*It is required to describe on $AB$ a segment of a ⊙ which shall contain an $\angle = \angle C$.*

At pt. $A$ in st. line $AB$ make $\angle BAD = \angle C$. I. 23.

Draw $AE \perp$ to $AD$, and bisect $AB$ in $F$.

From $F$ draw $FG \perp$ to $AB$, meeting $AE$ in $G$. Join $GB$.

Then in $\triangle$s $AGF$, $BGF$;

∵ $AF = BF$, and $FG$ is common, and $\angle AFG = \angle BFG$;

∴ $GA = GB$. I. 4.

With $G$ as centre and $GA$ as radius describe a ⊙ $ABH$.

Then will $AHB$ be the segment reqd.

For ∵ $AD$ is $\perp$ to $AE$, a line passing through the centre,

∴ $AD$ is a tangent to the ⊙ $ABH$. III. 16.

And ∵ the chord $AB$ is drawn from the pt. of contact $A$,

∴ $\angle BAD = \angle$ in segment $AHB$, III. 32.

that is, the segment $AHB$ contains an $\angle = \angle C$,

and it is described on $AB$, as was reqd.

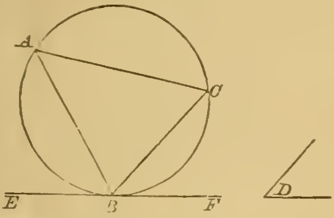
Q. E. F.

Ex. 1. Two circles intersect in $A$, and through $A$ is drawn a straight line meeting the circles again in $P$, $Q$. Prove that the angle between the tangents at $P$ and $Q$ is equal to the angle between the tangents at $A$.

Ex. 2. From two given points on the same side of a straight line, given in position, draw two straight lines which shall contain a given angle, and be terminated in the given line.

## Proposition XXXIV. Problem.

*To cut off a segment from a given circle, capable of containing an angle equal to a given angle.*



Let *ABC* be the given ⊙, and *D* the given ∠.

*It is required to cut off from ⊙ ABC a segment capable of containing an ∠ = ∠ D.*

Draw the st. line *EBF* to touch the circle at *B*.

At *B* make ∠ *FBC* = ∠ *D*.

Then ∵ the chord *BC* is drawn from the pt. of contact *B*,

∴ ∠ *FBC* = ∠ in segment *BAC*, III. 32.

that is, the segment *BAC* contains an ∠ = ∠ *D*;

and ∴ a segment has been cut off from the ⊙, as was reqd.

Q. E. F.

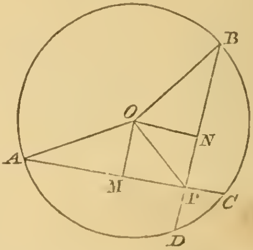
Ex. 1. If two circles touch internally at a point, any straight line passing through the point will divide the circles into segments, capable of containing equal angles.

Ex. 2. Given a side of a triangle, its vertical angle, and the radius of the circumscribing circle: construct the triangle.

Ex. 3. Given the base, vertical angle, and the perpendicular from the extremity of the base on the opposite side: construct the triangle.

## PROPOSITION XXXV. THEOREM.

*If two chords in a circle cut one another, the rectangle contained by the segments of one of them, is equal to the rectangle contained by the segments of the other.*



Let the chords $AC$, $BD$ in the ⊙ $ABCD$ intersect in the pt. $P$.

*Then must rect.* $AP$, $PC$ *= rect.* $BP$, $PD$.

From $O$, the centre, draw $OM$, $ON$ ⊥s to $AC$, $BD$,
and join $OA$, $OB$, $OP$.

Then ∵ $AC$ is divided equally in $M$ and unequally in $P$,
∴ rect. $AP$, $PC$ with sq. on $MP$ = sq. on $AM$. II. 5.
Adding to each the sq. on $MO$,
rect. $AP$, $PC$ with sqq. on $MP$, $MO$ = sqq. on $AM$, $MO$;
∴ rect. $AP$, $PC$ with sq. on $OP$ = sq. on $OA$. I. 47.

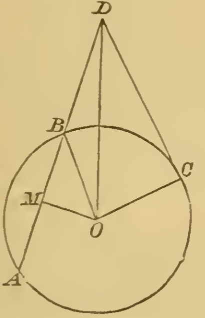
In the same way it may be shewn that
rect. $BP$, $PD$ with sq. on $OP$ = sq. on $OB$.
Then ∵ sq. on $OA$ = sq. on $OB$,
∴ rect. $AP$, $PC$ with sq. on $OP$ = rect. $BP$, $PD$ with sq. on $OP$;
∴ rect. $AP$, $PC$ = rect. $BP$, $PD$. Q. E. D.

Ex. 1. $A$ and $B$ are fixed points, and two circles are described passing through them; $PCQ$, $P'CQ'$ are chords of these circles intersecting in $C$, a point in $AB$; shew that the rectangle $CP$, $CQ$ is equal to the rectangle $CP'$, $CQ'$.

Ex. 2. If through any point in the common chord of two circles, which intersect one another, there be drawn any two other chords, one in each circle, their four extremities shall all lie in the circumference of a circle.

## Proposition XXXVI. Theorem.

*If, from any point without a circle, two straight lines be drawn, one of which cuts the circle, and the other touches it; the rectangle contained by the whole line which cuts the circle, and the part of it without the circle, must be equal to the square on the line which touches it.*



Let $D$ be any pt. without the ⊙ $ABC$,

and let the st. lines $DBA$, $DC$ be drawn to cut and touch the ⊙.

*Then must rect.* $AD$, $DB=$ *sq. on* $DC$.

From $O$, the centre, draw $OM$ bisecting $AB$ in $M$,

and join $OB$, $OC$, $OD$.

Then ∵ $AB$ is bisected in $M$ and produced to $D$,

∴ rect. $AD$, $DB$ with sq. on $MB=$sq. on $MD$. II. 6.

Adding to each the sq. on $MO$,

rect. $AD$, $DB$ with sqq. on $MB$, $MO=$sqq. on $MD$, $MO$.

Now the angles at $M$ and $C$ are rt. ∠s : III. 3 and 18.

∴ rect. $AD$, $DB$ with sq. on $OB=$sq. on $OD$;

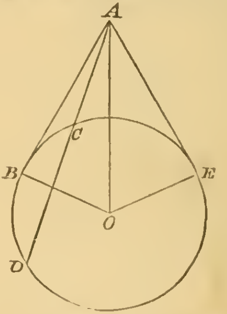
∴ rect. $AD$, $DB$ with sq. on $OB=$sqq. on $OC$, $DC$. I. 47.

And sq. on $OB=$sq. on $OC$;

∴ rect. $AD$, $DB=$sq. on $DC$. Q. E. D.

### Proposition XXXVII. Theorem.

*If, from a point without a circle, there be drawn two straight lines, one of which cuts the circle, and the other meets it: if the rectangle contained by the whole line which cuts the circle, and the part of it without the circle, be equal to the square on the line which meets it, the line which meets must touch the circle.*



Let $A$ be a pt. without the ⊙ $BCD$, of which $O$ is the centre.

From $A$ let two st. lines $ACD$, $AB$ be drawn, of which $ACD$ cuts the ⊙ and $AB$ meets it.

*Then if rect.* $DA, AC =$ *sq. on* $AB$, $AB$ *must touch the* ⊙.

Draw $AE$ touching the ⊙ in $E$, and join $OB$, $OA$, $OE$.

Then ∵ $ACD$ cuts the ⊙, and $AE$ touches it,
∴ rect. $DA, AC =$ sq. on $AE$. III. 36.
But rect. $DA, AC =$ sq. on $AB$; Hyp.
∴ sq. on $AB =$ sq. on $AE$;
∴ $AB = AE$.
Then in the △s $OAB$, $OAE$,
∵ $OB = OE$, and $OA$ is common, and $AB = AE$,
∴ ∠ $ABO =$ ∠ $AEO$. I. c.
But ∠ $AEO$ is a rt. ∠; III. 18.
∴ ∠ $ABO$ is a rt. ∠.
Now $BO$, if produced, is a diameter of the ⊙;
∴ $AB$ touches the ⊙. III. 16.

Q. E. D.

## *Miscellaneous Exercises on Book III.*

1. The segments, into which a circle is cut by any straight line, contain angles, whose difference is equal to the inclination to each other of the straight lines touching the circle at the extremities of the straight line which divides the circle.

2. If from the point in which a number of circles touch each other, a straight line be drawn cutting all the circles, shew that the lines which join the points of intersection in each circle with its centre will be all parallel.

3. From a point $Q$ in a circle, $QN$ is drawn perpendicular to a chord $PP'$, and $QM$ perpendicular to the tangent at $P$: shew that the triangles $NQP'$, $QPM$ are equiangular.

4. $AB$, $AC$ are chords of a circle, and $D$, $E$ are the middle points of their arcs. If $DE$ be joined, shew that it will cut off equal parts from $AB$, $AC$.

5. One angle of a quadrilateral figure inscribed in a circle is a right angle, and from the centre of the circle perpendiculars are drawn to the sides, shew that the sum of their squares is equal to twice the square of the radius.

6. $A$ is the extremity of the diameter of a circle, $O$ any point in the diameter. The chord which is bisected at $O$ subtends a greater or less angle at $A$ than any other chord through $O$, according as $O$ and $A$ are on the same or opposite sides of the centre.

7. If a straight line in a circle not passing through the centre be bisected by another and this by a third and so on, prove that the points of bisection continually approach the centre of the circle.

8. If a circle be described passing through the opposite angles of a parallelogram, and cutting the four sides, and the points of intersection be joined so as to form a hexagon, the straight lines thus drawn shall be parallel to each other.

9. If two circles touch each other externally and any third circle touch both, prove that the difference of the distances of

the centre of the third circle from the centres of the other two is invariable.

10. Draw two concentric circles, such that those chords of the outer circle, which touch the inner, may equal its diameter.

11. If the sides of a quadrilateral inscribed in a circle be bisected and the middle points of adjacent sides joined, the circles described about the triangles thus formed are all equal and all touch the original circle.

12. Draw a tangent to a circle which shall be parallel to a given finite straight line.

13. Describe a circle, which shall have a given radius, and its centre in a given straight line, and shall also touch another straight line, inclined at a given angle to the former.

14. Find a point in the diameter produced of a given circle, from which, if a tangent be drawn to the circle, it shall be equal to a given straight line.

15. Two equal circles intersect in the points $A$, $B$, and through $B$ a straight line $CBM$ is drawn cutting them again in $C$, $M$. Shew that if with centre $C$ and radius $BM$ a circle be described, it will cut the circle $ABC$ in a point $L$ such that arc $AL$=arc $AB$.

Shew also that $LB$ is the tangent at $B$.

16. $AB$ is any chord and $AC$ a tangent to a circle at $A$; $CDE$ a line cutting the circle in $D$ and $E$ and parallel to $AB$. Shew that the triangle $ACD$ is equiangular to the triangle $EAB$.

17. Two equal circles cut one another in the points $A$, $B$; $BC$ is a chord equal to $AB$; shew that $AC$ is a tangent to the other circle.

18. $A$, $B$ are two points; with centre $B$ describe a circle, such that its tangent from $A$ shall be equal to a given line.

19. The perpendiculars drawn from the angular points of a triangle to the opposite sides pass through the same point.

20. If perpendiculars be dropped from the angular points of a triangle on the opposite sides, shew that the sum of the squares on the sides of the triangle is equal to twice the sum of the rectangles, contained by the perpendiculars and that part of each intercepted between the angles of the triangles and the point of intersection of the perpendiculars.

21. When two circles intersect, their common chord bisects their common tangent.

22. Two circles intersect in $A$ and $B$. Two points $C$ and $D$ are taken on one of the circles; $CA$, $CB$ meet the other circle in $E$, $F$, and $DA$, $DB$ meet it in $G$, $H$: shew that $FG$ is parallel to $EH$.

23. $A$ and $B$ are fixed points, and two circles are described passing through them; $CP$, $CP'$ are drawn from a point $C$ on $AB$ produced, to touch the circles in $P$, $P'$; shew that $CP=CP'$.

24. From each angular point of a triangle a perpendicular is let fall upon the opposite side; prove that the rectangles contained by the segments, into which each perpendicular is divided by the point of intersection of the three, are equal to each other.

25. If from a point without a circle two equal straight lines be drawn to the circumference and produced, shew that they will be at the same distance from the centre.

26. Let $O$, $O'$ be the centres of two circles which cut each other in $A$, $A'$. Let $B$, $B'$ be two points, taken one on each circumference. Let $C$, $C'$ be the centres of the circles $BAB'$, $BA'B'$. Then prove that the angle $CBC'$ is the supplement of the angle $OA'O'$.

27. The common chord of two circles is produced to any point $P$; $PA$ touches one of the circles in $A$; $PBC$ is any chord of the other: shew that the circle which passes through $A$, $B$, $C$ touches the circle to which $PA$ is a tangent.

28. Given the base of a triangle, the vertical angle, and the length of the line drawn from the vertex to the middle point of the base: construct the triangle.

29. If a circle be described about the triangle $ABC$, and a straight line be drawn bisecting the angle $BAC$ and cutting the circle in $D$, shew that the angle $DCB$ will be equal to half the angle $BAC$.

30. If the line $AD$ bisect the angle $A$ in the triangle $ABC$, and $BD$ be drawn without the triangle making an angle with $BC$ equal to half the angle $BAC$, shew that a circle may be described about $ABCD$.

31. Two equal circles intersect in $A, B$: $PQT$ perpendicular to $AB$ meets it in $T$ and the circles in $P, Q$. $AP, BQ$ meet in $R$; $AQ, BP$ in $S$; prove that the angle $RTS$ is bisected by $TF$.

32. If the angle, contained by any side of a quadrilateral and the adjacent side produced, be equal to the opposite angle of the quadrilateral, prove that any side of the quadrilateral will subtend equal angles at the opposite angles of the quadrilateral.

33. If $DE$ be drawn parallel to the base $BC$ of a triangle $ABC$, prove that the circles described about the triangles $ABC$ and $ADE$ have a common tangent at $A$.

34. Describe a square equal to the difference of two given squares.

35. If tangents be drawn to a circle from any point without it, and a third line be drawn between the point and the centre of the circle, touching the circle, the perimeter of the triangle formed by the three tangents will be the same for all positions of the third point of contact.

36. If on the sides of any triangle as chords, circles be described, of which the segments external to the triangle contain angles respectively equal to the angles of a given triangle, those circles will intersect in a point.

37. Prove that if $ABC$ be a triangle inscribed in a circle, such that $BA = BC$, and $AA'$ be drawn parallel to $BC$, meeting the circle again in $A'$, and $A'B$ be joined cutting $AC$ in $E$, $BA$ touches the circle described about the triangle $AEA'$.

38. Describe a circle, cutting the sides of a given square, so that its circumference may be divided at the points of intersection into eight equal arcs.

39. $AB$ is the diameter of a semicircle, $D$ and $E$ any two points on its circumference. Shew that if the chords joining $A$ and $B$ with $D$ and $E$, either way, intersect in $F$ and $G$, the tangents at $D$ and $E$ meet in the middle point of the line $FG$, and that $FG$ produced is at right angles to $AB$.

40. Shew that the square on the tangent drawn from any point in the outer of two concentric circles to the inner equals the difference of the squares on the tangents, drawn from any point, without both circles, to the circles.

41. If from a point without a circle, two tangents $PT$, $PT'$, at right angles to one another, be drawn to touch the circle, and if from $T$ any chord $TQ$ be drawn, and from $T'$ a perpendicular $T'M$ be dropped on $TQ$, then $T'M = QM$.

42. Find the loci :

(1.) Of the centres of circles passing through two given points.

(2.) Of the middle points of a system of parallel chords in a circle.

(3.) Of points such that the difference of the distances of each from two given straight lines is equal to a given straight line.

(4.) Of the centres of circles touching a given line in a given point.

(5.) Of the middle points of chords in a circle that pass through a given point.

(6.) Of the centres of circles of given radius which touch a given circle.

(7.) Of the middle points of chords of equal length in a circle.

(8.) Of the middle points of the straight lines drawn from a given point to meet the circumference of a given circle.

43. If the base and vertical angle of a triangle be given, find the locus of the vertex.

44. A straight line remains parallel to itself while one of its extremities describes a circle. What is the locus of the other extremity ?

45. A ladder slips down between a vertical wall and a horizontal plane: what is the locus of its middle point?

46. *ABC* is a line drawn from a point *A*, without a circle, to meet the circumference in *B* and *C*. Tangents are drawn to the circle at *B* and *C* which meet in *D*. What is the locus of *D*?

47. The angular points *A*, *C* of a parallelogram *ABCD* move on two fixed straight lines *OA*, *OC*, whose inclination is equal to the angle *BCD*; shew that one of the points *B*. *D*, which is the more remote from *O*, will move on a fixed straight line passing through *O*.

48. On the line *AB* is described the segment of a circle in the circumference of which any point *C* is taken. If *AC*, *BC* be joined, and a point *P* taken in *AC* so that *CP* is equal to *CB*, find the locus of *P*.

49. The centre of the circle *CBED* is on the circumference of *ABD*. If from any point *A* the lines *ABC* and *AED* be drawn to cut the circles, the chord *BE* is parallel to *CD*.

50. If a parallelogram be described having the diameter of a given circle for one of its sides, and the intersection of its diagonals on the circumference, shew that the extremity of each of the diagonals moves on the circumference of another circle of double the diameter of the first.

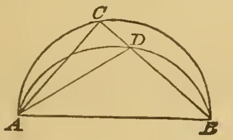
51. One diagonal of a quadrilateral inscribed in a circle is fixed, and the other of constant length. Shew that the sides will meet, if produced, on the circumferences of two fixed circles.

We here insert Euclid's proofs of Props. 23, 24 of Book III. first observing that he gives the following definition of similar segments :—

Def. *Similar segments of circles are those in which the angles are equal, or which contain equal angles.*

## Proposition XXIII. Theorem.

*Upon the same straight line, and upon the same side of it, there cannot be two similar segments of circles, not coinciding with each other.*



If it be possible, on the same base $AB$, and on the same side of it, let there be two similar segments of ⊙s, $ABC$, $ABD$, which do not coincide.

Because ⊙ $ADB$ cuts ⊙ $ACB$ in pts. $A$ and $B$, they cannot cut one another in any other pt., and ∴ one of the segments must fall within the other. III 10

Let $ADB$ fall within $ACB$.

Draw the st. line $BDC$ and join $CA$, $DA$.

Then ∵ segment $ADB$ is similar to segment $ACB$,

$$\therefore \angle ADB = \angle ACB.$$

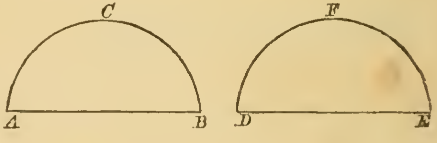
Or the extr. ∠ of a △ = the intr. and opposite ∠, which is impossible ; I 16

∴ the segments cannot but coincide.

Q. E. D.

## PROPOSITION XXIV. THEOREM.

*Similar segments of circles, upon equal straight lines, are equal to one another.*



Let $ABC$, $DEF$ be similar segments of ⊙s on equal st. lines $AB$, $DE$.

*Then must segment* $ABC$=*segment* $DEF$.

For if segment $ABC$ be applied to segment $DEF$, so that $A$ may be on $D$ and $AB$ on $DE$, then $B$ will coincide with $E$, and $AB$ with $DE$;

∴ segment $ABC$ must also coincide with segment $DEF$; III. 23.

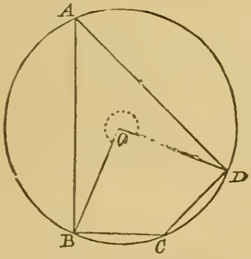
∴ segment $ABC$=segment $DEF$. Ax. 8.

Q. E. D.

We gave one Proposition, C, page 150, as an example of the way in which the conceptions of Flat and Reflex Angles may be employed to extend and simplify Euclid's proofs. We here give the proofs, based on the same conceptions, of the important propositions XXII. and XXXI.

### PROPOSITION XXII. THEOREM.

*The opposite angles of any quadrilateral figure, inscribed in a circle, are together equal to two right angles.*



Let $ABCD$ be a quadrilateral fig. inscribed in a ⊙.

*Then must each pair of its opposite ∠ s be together equal to two rt. ∠ s.*

From $O$, the centre, draw $OB$, $OD$.

Then ∵ ∠ $BOD$=twice ∠ $BAD$, III. 20.

and the reflex ∠ $DOB$=twice ∠ $BCD$, III. C. p. 150.

∴ sum of ∠ s at $O$=twice sum of ∠ s $BAD$, $BCD$.

But sum of ∠ s at $O$=4 right ∠ s ; I. 15, Cor. 2.

∴ twice sum of ∠ s $BAD$, $BCD$=4 right ∠ s ;

∴ sum of ∠ s $BAD$, $BCD$=two right ∠ s.

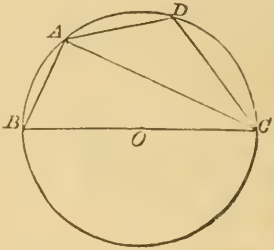
Similarly, it may be shewn that

sum of ∠ s $ABC$, $ADC$=two right ∠ s.

Q. E. D.

### PROPOSITION XXXI. THEOREM.

*In a circle, the angle in a semicircle is a right angle: and the angle in a segment greater than a semicircle is less than a right angle; and the angle in a segment less than a semicircle is greater than a right angle.*



Let $ABC$ be a $\odot$, of which $O$ is the centre and $BC$ a diameter.

Draw $AC$, dividing the $\odot$ into the segments $ABC$, $ADC$.

Join $BA$, $AD$, $DC$.

*Then must the $\angle$ in the semicircle $BAC$ be a rt. $\angle$, and $\angle$ in segment $ABC$, greater than a semicircle, less than a rt. $\angle$, and $\angle$ in segment $ADC$, less than a semicircle, greater than a rt. $\angle$.*

First, $\because$ the flat angle $BOC$ = twice $\angle BAC$, III. C p. 150.

$\therefore$ $\angle BAC$ is a rt. $\angle$.

Next, $\because$ $\angle BAC$ is a rt. $\angle$,

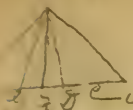
$\therefore$ $\angle ABC$ is less than a rt. $\angle$. I. 17.

Lastly, $\because$ sum of $\angle$s $ABC$, $ADC$ = two rt. $\angle$s, III. 22.

and $\angle ABC$ is less than a rt. $\angle$,

$\therefore$ $\angle ADC$ is greater than a rt. $\angle$.

Q. E. D.



[illegible] angle at A & AE [illegible]

AD on BC [illegible] ∠ FAE = [illegible]

∠: B, F & C [illegible] ⊥ [illegible] I 32

∠ FAB = ∠ FAC [illegible]

∠ FAD = ∠ FAE [illegible] ∠ FAC

[illegible] ∠ DAB [illegible] ∠

∠ B + ∠ BAD = rt ∠ I 32

∠ C + ∠ CAD = " " I 32

∠ B + ∠ BAD = ∠ C + ∠ CAD [illegible] I

∴ Diff. between ∠ B & ∠ C = [illegible] ∠ [illegible]

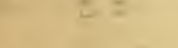
[illegible] ∠ BAD [illegible] (a [illegible]) ∠ [illegible]

= " ∠ EAD = [illegible] (angle ∠ FAD [illegible])

∴ ∠ FAD = ½ diff. between ∠ B & ∠ C

---

[illegible]



AD [illegible] BC + CE [illegible]

AC . CB = [illegible]

(AC + CB)(AC − CB) = [illegible]² − [illegible]²

AC . C [illegible]

[illegible]

(BC + AC)(BC − AC) = [illegible]² − [illegible]